DOCUMENT RESUME

ED 353 439 CE 062 868

AUTHOR Thiering, Jeannette; And Others
TITLE Teaching Vocational Mathematics.
INSTITUTION National Centre for Vocational Education Research, Leabrook (Australia).
REPORT NO ISBN-0-86397-246-2
PUB DATE 92
NOTE 71p.
AVAILABLE FROM National Centre for Vocational Education Research, 252 Kensington Road, Leabrook, South Australia 5068, Australia.
PUB TYPE Guides - Classroom Use - Teaching Guides (For Teacher) (052)

EDRS PRICE MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS Educational Strategies; Foreign Countries; *Mathematical Applications; Mathematical Formulas; *Mathematics Anxiety; *Mathematics Instruction; Postsecondary Education; *Problem Solving; Secondary Education; Teacher Education; *Vocational Education
IDENTIFIERS Australia; *Job Related Mathematics

ABSTRACT

This Australian document is a guide to teaching mathematics as it relates to specific work situations. After a brief introduction, chapter 1 looks at problem solving, advises teachers to lower the reading level and thus raise the understandability of written math problems, and describes a four-step problem-solving process. Chapter 2 outlines a lesson that teaches a math concept and provides other examples of using concrete aids. Chapter 3 discusses teaching estimation by using number skills, mathematical common sense, and concepts of size. The chapter also includes brief sections on practices in the trades regarding rounding numbers up and down. Chapter 4 covers formulas, including the terminology of formulas, their place in school mathematics, and how formulas are taught in vocational courses. Chapter 5 covers decimals, including the following: (1) the meaning of decimal notation; (2) the use of zero in decimals; (3) essential zeros; (4) the order of size in decimals; (5) rounding; (6) multiplying and dividing by 10, 100, and 1,000; and (7) multiplication and division by numbers smaller than 1. Chapter 6 covers common fraction ideas, equivalent fractions, and simplifying fractions. Chapter 7 deals with percentages by examining the following: (1) common percentage ideas; (2) comparing two quantities as a percentage; (3) decimals and percentages; (4) finding a percentage of a quantity; and (5) working out 100 percent when given another percentage. Chapter 8 offers worked examples of math problems relating to fitting and machining, plumbing or building, automotive engineering, and commercial cookery. Several chapters include student worksheets reinforcing the chapter concepts and information. (CML)

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Teaching Vocational Mathematics

JEANNETTE THIERING
SUE HATHERLY
JANICE McLEOD

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Teaching Vocational Mathematics

JEANNETTE THIERING
SUE HATHERLY
JANICE McLEOD

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Adelaide 1992

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A.C.N. 007 967 311

ISBN: 0 86397 246 2 (Hard copy)
TD/TNC: 31.26

Word processing by: Eleanor Woods and Kaye McAdam

Published by:
National Centre for Vocational Education Research Ltd,
252 Kensington Road,
LEABROOK SA 5068

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Introduction

Vocational mathematics differs from the abstract concepts and the hierarchy of mathematical skills which we are familiar with in general education. Mathematics is included in vocational courses because it is needed for specific work situations —there will always be some application of mathematics that is used on the job. Students' interest and motivation to learn will result from relating the maths to their current work or employment goals. Many practical courses have acknowledged this by developing integrated syllabuses so that mathematics is no longer taught separately. Instead, particular mathematical concepts or skills are taught during the course as they are needed. This book covers mathematical topics common to a variety of vocational contexts, in a straightforward, practical way.

The goal of vocational mathematics is to train people to work out solutions to problems they will meet on the job. This is why we begin with an approach to problem solving which will give students a framework to follow. Then we look at particular skill areas in which students are known to have difficulties. The last chapter shows five problems from popular courses being solved, using the recommended methods.

One of our underlying principles is that mathematics should be taught for understanding, using a variety of approaches including hands-on material. If students' learning is based on understanding, if they know why they are doing something, they have a much better chance of remembering the steps correctly. For example, if they have forgotten a formula, they should be able to work out the problem by drawing on their understanding of the processes involved.

Calculation skills should be taught and assessed in vocational courses as they come up in practical situations—never in isolation. We assume that calculators are accepted as an effective tool for problem solving and for doing calculations, but we also give considerable attention to developing in students a thinking approach to mathematics so that they can be intelligent users of their calculators. Commitment to calculator use, and to our other teaching methods, is prompted by the need for students to be taught in a way that gives continuity between vocational mathematics and the mathematics they studied at school.

We hope that all teachers involved in vocational mathematics will find this material relevant and useful.

A large proportion of *Teaching Vocational Mathematics* was originally published in *Trade Mathematics: A Handbook for Teachers*, (Thiering J, McLeod J and Hatherly S, Nelson Wadsworth 1987) which is out of print. All the earlier material has been revised for this book.

Jeannette Thiering
Sue Hatherly
Janice McLeod

1 : Problem solving

One of the great advantages of teaching mathematics as part of a vocational course is that the maths has a practical application. It may be the first time that students can see the relevance of maths; that it can be a tool for solving real problems. Therefore an integral part of the approach to teaching vocational maths should be problem solving. If students have negative attitudes to maths, and have never been able to see the point of it, linking maths to its application in the workplace will help overcome this barrier and motivate them to learn.

Many students seem unable to apply their mathematical skills and practical knowledge to written calculations. They either take one look at a written problem and decide they can't do it, or start working with the first few numbers they come across, without fully reading the question.

Many teachers experience varying degrees of concern, frustration and annoyance when their students, having successfully tackled a practical problem in the workshop, seem prepared to accept impossibly large or small answers to related written calculations, or hand-in pages of muddled working.

It is important to try and analyse the reason for such student difficulties with written calculations and not just to assume that 'it's their maths'. It may well be a mathematical difficulty, since problem solving is a real test of students' understanding of the mathematical concepts underlying their work. It may be, however, that the students, especially second-language learners, cannot even get to a point where they can *begin* to apply mathematical skills. There may be words or phrases in the question that they cannot read. It could be that they can read the words, but do not fully understand what the question means.

Writing problems

Before helping students develop strategies for solving written calculations, the way the questions are written needs to be looked at. Are they clear and easy to understand, or are they full of complicated terms and phrases and unnecessary or confusing information? Could they be rewritten in a better way?

The following points are just some to be aware of when you are writing or selecting written calculations.

a Uncommon or difficult-to-read words can often be replaced by easier words without changing the meaning of the question.

For example	*Possible rewrite*
1 Determine the missing values in the table relating to spirals...	*Work out the missing values in the table about spirals ...*

For example	*Possible rewrite*
2 Calculate the diameter in mm to turn the end prior to milling the square.	*Work out the diameter in mm to turn the end before milling the square.*
3 Two metres in length...	*Two metres long ...*
4 A plumber is required to construct...	*A plumber has to build ...*
5 What is the force required to maintain motion of a trailer?	*What is the force needed to keep the trailer moving?*
6 What is the distance moved by the clutch throwout bearing...?	*What distance does the clutch throwout bearing move ...?*
7 How many metres of timber should be purchased...?	*How may metres of timber should you buy ...?*

b Difficult or long-winded phrases can make questions unnecessarily complicated. The use of pronouns and the present tense makes questions more personal and easier to read. This also helps students to see themselves actually doing the work, e.g. adjusting the tappets, checking an angle with a sine bar, digging the trench.

The changes suggested do not necessarily make the sentences shorter. Sometimes more words are needed to make a point clearly.

For example	*Possible rewrite*
1 It is specified that 100 × 75 hardwood bearers be used in a floor 6m × 4.8m...	*You are laying a floor 6m × 4.8m using 100 × 75 hardwood bearers ...*
2 If it should be required to take into consideration a cutting waste of 15%...	*If you need to allow a cutting waste of 15% ...*
3 Speed variations are obtained with stepped pulleys...	*Stepped pulleys vary the speed ...*
4 The following components were purchased by a mechanic...	*A mechanic bought the following parts ...*
5 Calculate the amount of expansion which would occur in a copper pipeline 17m in length if the temperature of the contained water increased from 15°C to 78°C.	*How much would a copper pipeline 17m long expand if the temperature of the water in it increased from 15° to 78°?*
6 A water tank is required to hold 37 500 litres and is restricted to a length of 5m and 3.75m in width. What would be the height of the tank?	*A water tank has to hold 37 500 litres and can only be 5m long and 3.75m wide. How high would the tank have to be?*
7 Using sketches as aids determine...	*Draw sketches and use them to help you work out ...*

For example	*Possible rewrite*
8 Determine the angle to set the compound rest to cut the taper illustrated...	*You have to cut the taper in the drawing. Work out what angle to set the compound rest at ...*
9 A trade discount of 20% is to be allowed...	*Allow a trade discount of 20% ...*
10 What is the power consumption of a circuit if... ?	*How much power would a circuit use if ...?*

c Many mathematical or technical terms have different meanings in regular English usage. Check that students know the precise mathematical meaning.

For example	
The *volume* of a cylinder...	The *volume* on a radio...
What is the *area* of a template?	Which *area* do you live in?
The circular *pitch* of a spur gear...	The cricket *pitch*...
The engine *developed* power.	You get a film *developed*.

d There are many different ways of asking the same type of question. Make sure your students understand this varying phraseology.

For example

Area calculations can be written as:

- find the area of...
- what is the surface area in m^2 of...
- calculate the total friction area of...
- find the cost of painting the outside of...
- what is the cross-sectional area of...
- find the road surface contact area of...
- calculate the total opening in an engine cylinder which has...
- how many square metres of covering...
- estimate the number of m^2 of tiling needed...
- what is the total bolt area under stress...
- what is the catchment area of...

Volume calculations can be written as:

- calculate the cubic capacity of an engine...
- calculate the engine displacement in cc and litres of a...
- calculate the swept volume of...
- find the volume of a footing...
- how many cubic metres of concrete will you need...
- calculate the quantity of excavated material...
- estimate the cubic contents of...
- calculate the amount of filling required...
- how many litres of water will a tank hold...
- how many m^3 of soil must be carted away...

e The question may lack certain details needed for a solution. Teachers often assume knowledge which their students do not have. As a check always proof-read your own questions and ask someone else, preferably from outside your area to read them through for you. They will often pick up ambiguities or other problems in question format that you have not noticed.

In general try to write questions as you would ask them in the workplace. If, however, you know that examination questions are generally written in a formal way, you will need to teach students this kind of language.

The problem-solving process

Problem solving involves a lot of decision making. You have to decide what the question means, what information you need and what you don't need, what to do first, which operation to use and whether your final answer is reasonable. To complicate the process further you may make a careless computational error somewhere along the way or worse still press the wrong button on your calculator.

Many students feel they can't do problem solving. As teachers you need to build up student confidence about problems and convince them that not knowing doesn't mean failing; it just means not having found out yet. You can teach strategies and ways of attaching a problem, which can help students to develop their own problem-solving abilities. By exposing students to a range of different problems and focussing on the process of finding the solution (not just the answer itself), you can help your students develop skills and flexibility for the practical demands of the work situation where no two 'problems' are ever exactly the same.

Properly used, the calculator can be a confidence builder and can increase the problem-solving efficiency of its user. Calculators do not solve problems, people do, but they can free students from long and cumbersome calculations and allow them to concentrate on the more important aspects of the problem-solving process.

There are four basic steps in the problem-solving process:

- get to know the problem
- choose what to do
- find a solution
- look back

These are the steps that good problem solvers go through, the steps that people who are good at maths use. They are the steps needed to get the right answer, not just an answer that may or may not be right. The process is necessary for successful problem solving and should be taught.

Teaching problem solving

Teaching students the four steps in problem solving gives them a useful structure for starting the problem-solving process. Students need a lot of help and many examples to work through before they become confident problem solvers. It would be useful

to include a unit of work aimed at developing problem-solving skills in the curriculum for the early weeks of a course. As vocational teachers, you have many excellent opportunities for teaching problem solving, using genuine problems with real-life numbers.

Each step can be taught in different ways.

Step 1: Get to know the problem

READING FOR UNDERSTANDING

Reading mathematical material is different from reading narrative prose. When we read a newspaper or a novel we tend to read quickly, with only slight attention to detail. We are concerned with the overall meaning of the passage.

In comparison, written mathematical problems contain a lot of information in a few words, often with technical vocabulary and symbols. You may need to read tables, charts and graphs as well as interpret ratios, scales, formulas and drawings. You should read such material slowly and carefully and be prepared to re-read it several times to make sure you really understand what you have to find out.

The best way to check students' understanding of the concepts and vocabulary in a question is for them to rewrite the problem in their own words. Let them discuss the meaning of the problem in pairs or groups; by using and rephrasing the mathematical language, they will gradually come to understand it.

Teachers could also:

- let the students work through a set of problems, writing down what information they know and what extra information they need to know to work out a solution;
- give students a page of written calculations and let them work through them, noting information they do not need to use. Students often think they must use all the numbers mentioned in a problem.

Sorting information is an important aspect of problem solving, as real-life problems always contain extra information. For example, give students problems like the following:

> A vehicle uses 56 litres of fuel on the forward journey and 48 litres on the return journey. If the single journey was 680km and the fuel cost 74.7 cents per litre, what was the total cost of fuel for the complete trip?

In this example the distance is unnecessary information.

> Work out the stress in a drive belt with dimensions 50 x 5 x 300mm. The force applied to the belt is 900N.

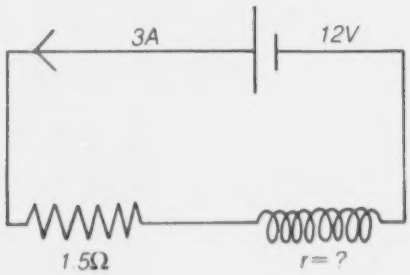
Here you don't need to know the thickness of the belt to work out the answer.

USING A DIAGRAM

Drawing a diagram or rough sketch can help students imagine a practical situation and increase their understanding of the problem. The diagram should show the relative sizes involved and any other important information.

> A 12 volt ignition circuit has a 1.5 ohm resistor connected in series with the coil. The current flow in the circuit is a 3 amperes. Calculate the resistance of the coil.

At first reading, this question could be confusing to students who are used to using the formula $R = \frac{E}{I}$ because the question gives a value for all three letters of the formula. Drawing and labelling the diagram helps the student to see what has to be done.



- The resistance given is for only one resistor. It is the resistance in the coil that is needed.

Step 2: Choose what to do

Choosing a method is perhaps the most difficult part of the problem-solving process. Teachers need to encourage students to focus on the thinking involved, not just the answer. Emphasise persistence, not speed. Let students discuss in pairs or groups different ways of solving the problem—which operations they will need to use and the different steps they will need to work through before reaching a solution. Encourage them to think about previous problems like this and remember what they did then. Help them to look for connections between the information they have and the results they want. The question really is 'How do I get from what I know to what I need to know?'

One strategy is to suggest the students replace the original difficult numbers with easy ones and imagine they are doing the activities themselves.

> *For example:*
> A car travels 195 kilometres and takes 2 hours 15 minutes to reach its destination. Find the average speed in km/h.

Say to the student:
If you drove 200km and it took you two hours, how fast were you going?

Most students can do this problem mentally and they will say '100km/h'. The answer makes sense to them. The next step is to ask, 'How did you get 100km/h?' If they don't know, repeat the process with other easy numbers until they realise that they divided the 200 by the 2, i.e. divided the distance by the time.

Students may have difficulty in deciding what operation to use for a written problem. Examining the links between language and basic operations can help. Ask students to list the ways you can be asked to add, subtract, multiply and divide. Typical lists will look like this:

- \+ add, sum, total, how many altogether?
- × lots of, of, how many altogether?
- – less than, less, off, reduce by, difference between
- ÷ divide up, split up, out of, how many in, reduce by, share, how many can you get out of . . .?

Students are often confused about when to multiply and when to divide—particularly when working with numbers less than one. Suggest to students they think about whether the question includes, or can be rephrased to include, the concept of 'of' or 'lot of' which can then be replaced by '×'. For example, when a question includes halving a quantity, because students connect half with division by 2 they want to divide by 0.5. Emphasise the language of the problem and how 'half of' means '½ ×' or 0.5 ×'. This may help them overcome the problem.

When choosing what to do, students also have to decide:

- which formula to use, e.g. πr^2 or $\frac{\pi d^2}{4}$ for the area of a circle;
- when to convert metric units, e.g. mm to m;
- which parts of the calculation to do first;
- how to lay out the problem;
- what a reasonable answer to the calculation might be.

For example:

A six-cylinder engine has a bore of 86mm and stroke of 75mm. Work out the engine displacement.

To choose what to do, the student has to:

- Know that engine displacement is equivalent to the volume of all the cylinders.
- Remember the formula $V = \frac{\pi d^2}{4}h$
- Know that the bore is the diameter (d) and the stroke is the height (h).
- Remember to multiply the volume of one cylinder by 6 to get the engine displacement.
- Realise that it will be easier to change mm to cm before substituting in the formula so that you get the answer in cubic centimetres.
- Divide the cc answer by 1000 to get the answer in litres.

Asking students to list all the steps involved, as in the above example, without working out the problem, gives them practice in this aspect of problem solving.

Step 3: Find a solution

Carrying out what you have decided to do is often a routine, computational process. The calculator is a good tool at this point. Many students believe that this is all that problem solving is about—the number crunching. It is important to convince them that it is only a small part of the process.

Before starting to do the calculation you should first get a rough idea of the answer. This allows you to check the actual answer you work out and gives you a safeguard against errors. Ways of developing the skill of estimating are discussed in chapter 3. If teachers want their students to make estimates, they should always include an estimate when working problems on the board or with students individually. If the problem is not seeking a numerical result, or if it is an open-ended question or one with more than a single solution, then making an estimate will not be appropriate.

Step 4: Look back

This is the stage of problem solving that is most often neglected. When students solve a problem or arrive at an answer, they often think their job is finished. Even if they have made an estimate, there is no guarantee that they will actually go back and compare it with their answer. It is, however, the step that takes them closest to the adult attitude of aiming for 100% accuracy. It means that the problem solver is not satisfied with any answer. He or she wants the *right* answer.

Suggesting that they always write out an answer using the words of the question will help make sure that they look back. They also need to:

- go back and check that they have answered all parts of the question;
- check that the solution satisfies all the given conditions;
- compare their answer with their estimate; and
- ask whether the answer is reasonable.

This last point is particularly important. If students are giving ridiculous answers such as 250 cubic metres of concrete to pour a small garage floor, it may be because they simply haven't asked themselves whether it really could be that big. They need to be encouraged to listen to their brain saying 'That can't be right!' and work through the problem again. It helps if they can work with another student to discuss their solution. Often explaining to someone else what you did will help you see your mistakes.

It is important that problems always use realistic figures such as up-to-date prices and sensible measurements so that students can use their knowledge of the real world when checking their answers. Strategies to teach concepts of size will be discussed in chapter 3. Clear concepts of size will help students imagine what their answers mean. They will be able to answer questions such as: Would the bolt be that small? Would you need that much concrete to lay a path? Would it really cost that much? If they are familiar with average sizes and recent prices they will be better able to make these judgments. If the students are not employed in the field they are studying, their only access to this information will be from their teachers.

Worksheet 1 is a suggestion for giving students practice in checking the reasonableness of answers. The idea is that students do no working out. They should just think about the alternative answers and decide which one is most likely to be right. It would be even better to make up a similar job-specific exercise. As with all the other stages of the problem-solving process, teachers should always discuss the reasonableness of the answer with the students when they are working a problem on the board.

The thinking approach

The key to successful problem solving lies in thinking about what you are doing at every step. Even when performing routine aspects such as straight computation, students should be encouraged to think about what it means; e.g. when adding 0.5 and 0.7, students should be encouraged to think 'A half plus more than a half means the answer should be greater than one'.

An important element in achieving a thinking approach in your students is lots of teacher–student and student–student discussion, based on concrete examples—something that will have been lacking in many people's maths education. Traditionally maths was taught abstractly, using symbols written on the board, with almost no links to real life and very little explanation in everyday language. No wonder many people think maths is hard!

If you are very familiar with something, it is relatively easy to explain it using specialist terms. The real test of whether you understand something, however, is your ability to explain it in clear and straightforward language. Encourage students to talk to you and to each other and listen carefully for the levels of understanding in words they use.

The questions you ask in class will be important in leading your students to think about what they are doing. Getting students to talk about what they are doing, and why, is the best way of making them think—and thinking is the key to good problem solving. Ask students questions like: What does this problem really mean? Why do we do it this way? Is there any other way to do it? How does this link to the way we do . . .? Now, tell me in your own words how we . . .?

Another important aspect of good maths teaching is the feedback students get on their written calculations. If you are working over problems in class, hold back from giving the correct answer too soon and, instead, suggest clues that will help them to work it out for themselves. If you are marking written work, ticks and crosses alone won't really explain to students why they went wrong. If you can't talk about it with the students individually, make written comments on their attempted solutions. They need to be encouraged to keep working at a problem until they get it right. Getting a wrong answer should only mean that they haven't got it right *yet*.

Worksheet 1

CHECKING THE REASONABLENESS OF RESULTS

Which one of each set of answers is most likely to be correct?

1 My car's petrol tank holds:

15L
50L
200L

2 The amount of concrete needed for the garage floor will be:

$2m^3$
$0.2m^3$
$12m^3$

3 The new-born baby weighs:

7.3kg
3.7kg
0.735kg

4 The area of this circle is:

$94mm^2$
$150mm^2$
$707mm^2$

5 The height of the prison escapee is:

140cm
170cm
240cm

Fold under

ANSWERS

***1**. 50L; **2**. $2m^3$; **3**. 3.7kg; **4**. $707m^2$; **5**. 170cm*

Teaching Vocational Mathematics, Thiering, J, McLeod, J and Hatherly, S, NCVER 1992.

2 : Teaching concepts

Why should we bother to teach concepts? Why not just teach the processes and the rules? The answer is that we teach concepts because knowledge is based on concepts. If the concepts have been grasped, the rules and processes will be:

- understood
- applied correctly
- remembered

The aim of vocational teaching is to produce practical people who know their jobs. To know something in a trade context means that you learnt it well in the first place and practise it often so that you remember it. Workers apply their knowledge to routine tasks and adapt it as circumstances change. If something is well understood from the beginning then you know:

- what it's about
- what it all means
- why it's done this way
- how it works

Teaching concepts thoroughly requires a multi-sensory approach. Students should be given experiences that involve far more than abstract thinking with words, diagrams, numbers, formulas and other symbols. Learning new concepts involves handling objects or models, feeling them, walking around to look at them from different directions, taking them apart and putting them back together. There are more senses operating than just sight and hearing. Students absorb ideas like size and mass, shape and pattern. Through these experiences they may also grasp the restrictions of what can and cannot be done.

If students are given a learning-by-doing experience, with a variety of activities including real objects and models, the chances are that their interest level will be very high and they will not forget that experience. The trouble is that the mathematical concept which you wanted them to learn may not be remembered. You have to help them to make the connections between the concrete experience and the way the concept is written down and used in calculations.

Learning moves from concrete objects to the use of symbols, so there have to be clear links between the learning experiences that introduce new concepts and the rules, formulas and calculations. The links begin to develop from the first time the concept is met and are transferred to the calculations only if the teacher is very careful at every step.

For example:

- Show a model. Talk about it. Get the students to discuss and handle it and ask questions. Include everyone.
- Without moving away from the model, write the formula and the calculations on some handy paper, e.g. on A3 paper placed beside the model, until all the links are clear.
- Take the paper to the board and record the formula and the calculations there. Keep the paper for future reference.
- Go back to the model. Ask students to explain each part of the board calculations by touching the real object.

- Give students calculations to do on their own paper, alone or in pairs and small groups. Meantime, move around the room to check for confusions until everyone can handle the process. Individual students can be shown the model or the paper again to strengthen understanding.
- When everyone can explain each feature of the calculation, change to symbols only and do more examples for reinforcement.

Later, there will be students who will forget and make errors. They can go back to the model and repeat steps. Concrete teaching aids have to be stored in a handy place so they can be used again and again quite easily. They need to be a sensible size and easily assembled. Some students will appreciate being able to revise the concept using the model, even months after they first changed to symbols.

Those students who missed out on the first lesson due to illness, say, should be allowed to work with the model until they can show that they have grasped the concepts. Otherwise they will blindly follow the rules for the calculations and be left wondering what it all means.

Quite often when the teacher uses a model to teach a topic, like hip roofs and gable roofs, some of the students will realise that, to them, it explains something quite different.

They will see that if it is turned upside-down it shows roof valleys, or that it could be easily changed to look like a trench with a trapezium cross-section. So the student performs the teacher's role for a short while and creative learning takes place.

Outline of a lesson teaching a concept

Rationale

Many students confuse area and perimeter. Lessons which strengthen the two concepts have a valuable carryover to practical applications. In this lesson we explore the relationships between area and perimeter when one of them is a fixed quantity.

Procedure

PART A

Step 1: Hand out to students four tiles and a sheet of grid paper. Ask them to arrange all four tiles on the desk so that a full side of each tile touches a full side of another. One result might be:

Step 2: Check the arrangements around the room to see that the tiles are touching correctly. Students should then sketch the shapes they have made on the grid paper.

Step 3: Repeat this until they cannot find any more shapes. Every student should be given time to find three for four shapes. Altogether this should yield five shapes around the class:

Step 4: Record all the shapes on the board or an OHP transparency. Have the students record all the shapes on the grid paper. Establish, by discussion, that all the shapes have the same area.

Step 5: Ask students to count around the shapes and write down the perimeter of each. One side of a square counts as one unit. (The answers are 10 units for four of the shapes, but the square's perimeter is eight units.)

Step 6: Pose the question: if different shapes have the same area, do they also have the same perimeter? Discuss this.

Step 7: Record on the board:

Shapes which have the same area do not all have the same perimeter.

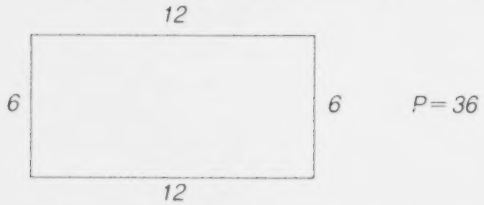
PART B

Give the students this problem to solve.

I have 36m of fencing to use around a vegetable garden in the middle of the backyard. I want to have a square or rectangular garden. What are the possible lengths and breadths I can use?
(Do not use parts of a metre.)

Materials needed: a piece of string about five metres long, sheets of 1cm grid paper or dot paper.

Step 1: Tie the ends of the string together and use two students to hold it so that it becomes a rectangular shape. This is a model of the 36 metres of fencing. Ask the students to sketch one shape on the grid paper. One possible shape might be:



Step 2: Check the answers around the room to see that no one is using numbers that multiply to give 36 by mistake, e.g. 9 and 4. Correct this error by discussion.

Step 3: Using grid paper, ask students to draw as many different shapes as they can to fit the problem. Every student should be able to find several shapes. Collect all answers around the class. The full range of possibilities is:

Length	Breadth
17	1
16	2
15	3
14	4

Length	Breadth
13	5
12	6
11	7
10	8
9	9

Step 4: List the answers in order on the board. Have the students draw all the shapes they missed on grid paper until everyone has nine sketches.

Step 5: Discuss the original situation: all the shapes use the same length of fencing. Ask students to record inside each sketch the area of the rectangle or square. Record these in a column beside the length and breadth columns already on the board.

Step 6: Ask the following questions:

- Do all the shapes have the same perimeter?
- Do all the shapes have the same area?

Discuss the results.

If time allows, patterns can be found in the sequence of answers in the area column. Reasons for this could be explored.

Step 7: Record on the board:

Shapes which have the same perimeter do not all have the same area.

PART C

Discuss the special case of the square. When the area was fixed the square had the smallest perimeter. When the perimeter was fixed the square had the largest area. Apply this to real-life situations like having the greatest space available for growing vegetables and designing economical house-plans.

This is a simple lesson. It can be presented in a short space of time.

On the other hand, it can take the form of an unusual homework assignment. See worksheet 2: Investigating areas and perimeters. Students' answers would be the basis of a discussion in the next lesson.

Other examples of using concrete aids

Squares and square roots

Students often confuse the concepts of squares and square roots, sometimes to the extent that if their calculator doesn't have a 'square' button they will use the square root button instead.

Small square tiles, all of equal size, can be useful to explain the difference. Starting with one tile, they can see that 1 is a square on its own. Later you can use this to show that the square and square root of 1 are also 1. Ask students to make the next largest square possible using tiles, and then the next and so on. When they have

arranged squares of at least four and nine tiles, ask them to count the tiles in each square and make a note of each total.

Then ask them what other number they can pick out in each arrangement:

- How many tiles in each row?
- How many tiles are along each side?

and, if necessary, ask them directly:

- What length are the sides of each square?

Point out that each square is based in a particular number, the square has a base, a root. Link the term 'root' to everyday meanings of the word—the root of a tree is at its base. If 9 is the square, 3 is the root, its 'base'. If you start with 3, you need 9 to make a square. Nine is the square of 3. The square number matches the square shape.

You can then introduce the symbols and have students record the facts. If you later need to move on from perfect squares, you can build on to the concept by adding that there will be many squares with sides larger than 2 but less than 3, i.e. squares between 4 and 9 and so on. For example, the square root of 6 will be a number between 2 and 3. Students can then confirm this using their calculators.

A similar approach can be taken to cubes and cube roots, using a number of small cubes to build larger cubes.

Factors

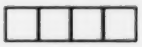
You can use the same small tiles to explain the meaning of factors. Ask students to take successively larger numbers of tiles and to make as many rectangles or squares as possible using the tiles. With two and three tiles there will be only one arrangement because they are prime numbers. With four there will be 1×4 and 2×2; with six, 1×6 and 2×3; but with twelve, there will be 1×12, 3×4 and 6×2. You can then link the idea of factors of a number being like the length and breadth of a rectangle whose area equals that number. In the patterns that emerge, they will also be able to see the difference between prime and composite numbers, and discover the numbers which are perfect squares.

It might be quicker simply to give a definition, but if students are first given time to explore concepts in ways like these, they will be more likely to remember them. They will associate many ideas with the concept and they will also remember the experience with the hands-on materials—all of which will help them recall the concept. That means you will save time later because repeated explanations will not be necessary.

Worksheet 2

INVESTIGATING AREAS AND PERIMETERS

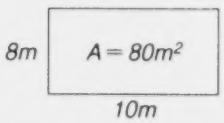
1 Take four square tiles or pieces of cardboard of any small size. Let the length of a side equal one unit. Arrange the four squares so that a full side of one touches a full side of another and find the perimeter.

e.g.  P = 10 units

Arrange them again in as many ways as you can.
Draw a sketch of each new shape you get and write under each sketch the perimeter, as above.
Keep going until you have five different shapes.

Questions: 1 Do all the shapes have the same area? .....
2 Do all the shapes have the same perimeter? .....
3 Which shape has the smallest perimeter? .....
4 Do all equal areas have the same perimeter? .....

2 If you had a piece of light fencing you could bend it to make a rectangle or a square. Imagine the fencing is 36m long and the sides have to be exact metre lengths, e.g. a rectangle, 10m long and 8m wide. Find the area in m^2 and write this inside the rectangle.



Find as many shapes as you can. Sketch each shape, label the lengths of the sides and calculate the area.

Questions: 1 Do all the shapes have the same perimeter? .....
2 Do all the shapes have the same area? .....
3 Which shape has the largest area? .....
4 Does every rectangle or square with perimeter 36m have the same area? .....

3 : Estimation

Estimation is an important life skill. The ability to estimate a length, area or quantity of material with a reasonable degree of accuracy is essential for anyone working in a vocational area.

The widespread use of calculators further increases the need for estimating. Calculator users need to develop a feeling for what the answer should be, and be alert for mistakes in key-stroking or malfunctions of their calculator. Worksheet 3.1 is a suggested way of giving students practice in actually using a calculator. No more details of how to use a calculator are given in this handbook. Secondary schools have made the use of a calculator a standard part of the syllabus. Good teaching materials can be found in school textbooks.

Estimating can be divided into three main areas:

- estimation using number skills
- estimation using mathematical commonsense
- estimation using concepts of size

Sometimes only one type of estimate can be made, but often two, or all three, are used in the same calculation.

Estimation skills can and should be taught. Students will only become confident estimators if they are given opportunities to practise estimating and to compare their estimates with exact answers or actual measurements.

Calculators will perform operations on numbers accurately, but the answer that the calculator gives may not be the right answer to the given problem. The student may have chosen the wrong operation, pressed the wrong key or missed out a step in the calculation. Before working out a calculation, students should always make an estimate of the size of the answer to be expected. This is especially important when working with decimals, as it helps to fix the position of the decimal point. Students should be encouraged to look critically at the answer that is displayed on the calculator to see if it is close to their estimate. In this way, estimating will reveal slips or careless errors.

Estimation using number skills

To estimate the results of a calculation before it goes on the calculator, students need to be able to do the following:

a Round off numbers to the nearest whole number, 10, 100, etc.

e.g. 287 rounded to the nearest 10 is 28$\overset{+1}{7}$. Answer: 290

b Handle multiplication and division with zeros.

e.g. 300 x 790, 1000 ÷ 50

Skills **a** and **b** are essential. Skills **c** and **d** are extra ones for 'smooth operators'.

c Choose the round numbers that make the estimating easy to do.

d Decide if the estimate is likely to be a bit bigger or a bit smaller than the exact answer.

Estimation worksheets 4, 5 and 6 were designed to help students develop the skills they need for estimation. Worksheet 4 is a pre-test that could be used to check what skills students already have. Worksheets 5 and 6 contain graded and fairly structured exercises designed to teach the component skills and to build confidence in using them. Worksheet 6 involves applying the skills developed in worksheet 5 to more complex calculations.

Comparing the estimate with the exact answer gives important feedback when learning to estimate. It proves that the process really works. In worksheet 6, however, the point is made that if obtaining an estimate is not possible or too difficult, students should be encouraged to check their work simply by repeating the calculation. This is meant to be a discussion worksheet where the processes are more important than the answers. If worksheets like these are used, it is a good idea to follow them immediately with some specific applied examples.

A great advantage of teaching students to do this kind of estimate before using their calculator is that they go through the steps of the problem, but with easy numbers. This should help students realise that there is no magic about the answer the calculator gives. Using a calculator is just a quicker way of working out a problem than doing it on paper.

Trade practices

Many trades have their own practices for rounding answers up, and occasionally down, at the end of a calculation.

- When buying timber, lengths have to be rounded up to the next multiple of 0.3m starting from a 1.8m length.
- Amounts of concrete have to be bought in multiples of $0.2m^3$.
- If you need 4.843 litres of paint, you will have to buy five litres.
- To find out how many 850mm lengths can be cut from a 6m length, you will need to round the answer to 6000 ÷ 850 down to 7.
- A shaft which has a diameter of 26.25mm, has a nominal diameter of 26mm.
- If you have a current of 8.33 amps, you need to round up to a 10 amp fuse.
- If a customer's bill comes to $43.792 you charge $43.80, or even perhaps $45!

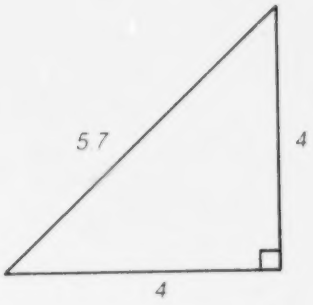
Estimation using mathematical commonsense

Sometimes it is possible to get a fairly accurate estimate by just considering the problem. Often the upper and lower limits for an answer are obvious.

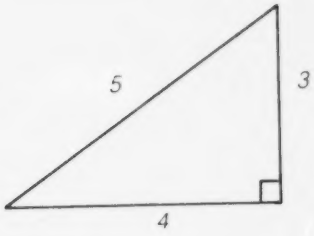
For example, when using Pythagoras' theorem to work out the length of the hypotenuse of a triangle, there is an absolute upper limit on the length of the hypotenuse in relation to the other two sides. The four examples below make the point.

The length of the hypotenuse is:

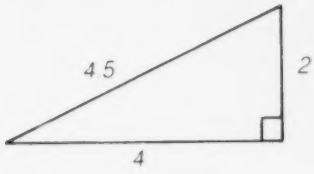
just less than
1½ × one of the other sides



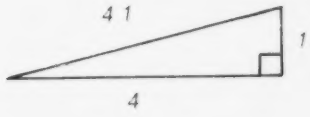
exactly
1¼ × the longer side



approximately
1⅛ × the longer side



equal to the longer
side + a bit



Two conclusions that can be made about the length of the hypotenuse are:

- The longest it can be is nearly 1½ × one of the other sides. This happens when the other two sides are equal.
- In a very thin triangle, it will only be a bit bigger than the longer side.

Using this knowledge, the student can easily work out the upper and lower limits for a reasonable answer and estimate a value which the hypotenuse will be close to.

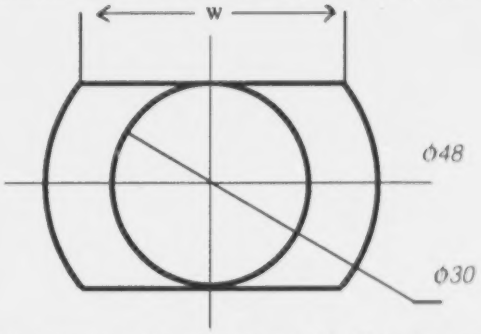
There are other obvious limits that an experienced person puts on his/her answer that less experienced students probably do not think of; for example, knowing that the area of a circle has to be less than the diameter squared or, even better, knowing it

will be approximately ¾ of the diameter squared. Applying the knowledge of what happens to the answer when dividing by a number less than 1 (see 'Decimals', pages 38–40) will also alert students to mistakes.

There are occasions when a fairly accurate estimate can be made by carefully looking at a diagram.

For example:
What is the width 'w' of the flats on the component shown in the diagram?

Note: ϕ 48 means the diameter is 48mm



Clearly the answer has to be between 30 (the shorter diameter) and 48 (the longer diameter).

In these cases, there is often no need to make the kind of estimate outlined in section 1 of this chapter.

The best way to encourage students to use commonsense when estimating is to talk with them about it. Asking students how they would estimate items will probably reveal many different methods. In this way students can learn from each other—and teachers may pick up a few ideas too.

Estimation using concepts of size

When students are working out problems which involve sizes, an estimate can often be made by using their knowledge of sizes of common objects. However, in order to do this, they need to have clear concepts of size. Experienced trades people have developed the ability, for example, to look at a wall and estimate its area, to look at a tank and have a good idea of its capacity and so on. This ability comes from working constantly using the relevant measurements. Young students are unlikely to have this ability, particularly students who are not getting on-the-job experience. Concepts of size therefore should be taught.

The first step is to make sure students have a clear idea of the metric system. A good technique is to show them everyday things that will give them a clear image of the various units. These can then be used to work out estimates of actual objects. Some suggestions are:

Linear measurements

millimetre: Width of the tip of a ballpoint pen

centimetre: Width of a person's little finger, or finger nail

metre: Have the students use a metre ruler to measure one metre from the floor up the side of their body.

kilometre: Suggest students use the trip meter in a car to measure one kilometre from their home along a familiar route.

Square measurements

hectare: Two soccer-fields side by side

square metre: Two useful teaching aids are a cubic metre kit consisting of 12 one-metre rods with corner jointers, and a set of base-10 blocks made up of 10mm and 100mm cubes, and 100mm × 10mm rods and 100mm × 100mm squares (both 10mm thick). Making a square metre first allows students to see just how big it is. If square metres are a frequently used measurement, outlining a square metre on the wall will allow students to become familiar with it and will help them when they attempt to estimate areas.

parts of a square metre: If students are often asked to work out very small areas and to describe them in terms of square metres, it is a good idea to mark out typical areas within the square metre and label them. For example, place the 100mm square from the base-10 blocks inside the square metre and label its upper surface $0.01m^2$. Repeat with the 100mm × 10mm rod: $0.01m^2$, and the 10mm cube: $0.0001m^2$. You could them construct other shapes by combining the various pieces and calculating them as part of a square metre. By seeing how many decimal places are involved in a typical answer, they will learn to expect very small decimal answers for small areas.

When you think students can calculate the area of regular shapes, move on to the actual sizes they work with. Introduce real objects if possible and use the blocks of known area to help estimate the area of the real thing.

Note: Not many students will have had much practice in working with decimals of more than three places. Making the connection between the number of 10mm squares and the area expressed in square metres will reinforce some decimal concepts. There is a paradox here about decimal notation: a 10 × 10 mm square fits 10 000 times into a square metre and its area is $0.0001m^2$. Ten thousand, 10 000, has four zeros. One ten-thousandth (0.0001) has only three essential zeros, but it does have four decimal places. The connection to emphasise is *four* zeros, *four* decimal places.

Cubic measurements

cubic metre: Fully assembling the cubic metre kit gives students a clear idea of its surprisingly large size.

Again, if students have to work out very small volumes in terms of cubic metres, placing a 10mm cube inside a cubic metre and labelling it
0.000 001m^3 gives them a standard to work from.

One suggestion for explaining how far a cubic metre of concrete spreads is to build a cubic metre with ten 100mm-thick square metres of foam which can then be spread over the floor.

Relationship between capacity, volume and mass

A cubic metre kit and base-10 blocks are an excellent basis for explaining the relationships within the metric system. Many students have not realised how simple the system is and working with these aids helps it to make sense.

The 100mm cube inside an assembled cubic metre can be used to make many connections. For example, a litre of water has a mass of a kilogram and fills a 100mm cube. A litre of water poured inside a cubic metre will be 1mm deep. 1000 litres of water would fill it and the filled cubic metre would weigh a tonne. In some jobs, 100 litres, or 1 kilolitre is a commonly used unit.

Worksheet 7 could be used to summarise some of these relationships.

The 100mm cube is also useful in explaining units of pressure. Filled with water, it exerts a pressure of nearly 1kPa on a surface. Students can hold it so they can actually feel a pressure of 1kPa. (Force due to gravity must be taken as approximately 10N.)

Once the students have clear concepts of size they need practice in estimating and checking their estimate against actual measurements. They will gradually become confident in their ability to estimate. It will also help if they know the average size of things they commonly work with; for example, the floor area of an average bedroom, the height of an average door. When smaller objects are involved, such as a cylinder, labelling one with its cross-sectional area and capacity will give students a reminder of what a reasonable answer should be.

Of course students will only become skilled at estimating if teachers use realistic measurements when they write questions.

Worksheet 3

USING A CALCULATOR

EXPLANATION

This worksheet aims to teach students to use their calculator correctly. An unusual feature of the worksheet is that the answers are given next to the calculations so that the effort goes into getting the correct answer, rather than just getting *an* answer. If students are using their calculator incorrectly, they will know straight away; they will have to try again and again until they come up with the correct result.

Items 1-4 help pick up careless errors. Encourage students to glance at the display before pressing the next button. Check that they know the use of C and CE buttons.

A calculator can be used to allow students to grasp mathematical concepts for themselves. Items 5-8 help them understand the relationship between squares and square roots.

Rounding numbers can cause problems for some students. In item 11, they will have to be able to round the answer to an appropriate number of decimal places. Note that some calculators will give the answer as 0.16666667.

Students need to know order of operations to complete items 12-16. Item 16 in particular often causes problems. The most common error is to put into the calculator: $10 + \pi \times 62$. Because the answer is provided, students will know immediately that something is wrong, and will either solve it for themselves or ask for help.

Turn over for the worksheet on using a calculator.

Teaching Vocational Mathematics, Thiering, J, McLeod, J and Hatherly, S, NCVER 1992.

Worksheet 3

USING A CALCULATOR

Use your calculator to do these calculations.
Check your answer with the answer given.

			ANSWERS
1	10.21 + 12.4026	=	22.6126
2	18.264 – 4.021	=	14.243
3	17.26 x 13.5	=	233.01
4	733346.91 x 0.0007069	=	518.4029307
5	42^2	=	1764
6	$\sqrt{1764}$	=	42
7	0.025^2	=	6.25×10^{-4} or 0.000625
8	$\sqrt{0.0007243}$	=	0.026912822
9	Change $\frac{12}{32}$ to a decimal	=	0.375
10	Change $\frac{24}{5}$ to a decimal	=	4.8
11	Change $\frac{1}{6}$ to a decimal	=	0.1666666666
12	$\frac{2 \times 3^2}{6} =$	=	3
13	$\frac{(2 \times 3)^2}{6} =$	=	6
14	$\frac{300 \times 30}{75} =$	=	120
15	$\frac{\pi \times 0.021^2}{4} =$	=	3.4636059×10^{-4} or 0.00034636
16	$\frac{10}{\pi \times 62} =$	=	0.051340304

Teaching Vocational Mathematics, Thiering, J, McLeod, J and Hatherly, S, NCVER 1992.

Worksheet 4

ESTIMATION PRE-TEST

Choose the answer most likely to be correct for each calculation. Do not actually work it out.

1 18.76 + 131.24 =

- **a** 212
- **b** 150
- **c** 128

2 693.07 – 124.93

- **a** 397.14
- **b** 818.21
- **c** 568.14

3 12½% of $489.50 =

- **a** $61.20
- **b** $6.12
- **c** $40.80

4 5.7^2 =

- **a** 11.4
- **b** 324.9
- **c** 32.49

5 5.5 × 2.64 =

- **a** 145.20
- **b** 14.52
- **c** 18.520

6 200 × 0.760 =

- **a** 152
- **b** 1520
- **c** 15.2

7 3 ÷ 0.6 =

- **a** 0.5
- **b** 50
- **c** 5

8 $\frac{175 \times 1.6 \times 60}{90}$ =

- **a** 186.67
- **b** 1866.7
- **c** 18.67

Fold under

ANSWERS

1	150	**5**	14.52
2	568.14	**6**	152
3	$61.20	**7**	5
4	32.49	**8**	186.67

Teaching Vocational Mathematics, Thiering, J, McLeod, J and Hatherly, S, NCVER 1992.

Worksheet 5

ESTIMATION EXERCISES 1

1 Round off these numbers to the nearest whole number.

7.2	12.6	0.8	19.5	49.7

2 Round off these numbers to the nearest 10.

27	35	512	7389	4306

3 Round off these numbers to the nearest 100.

423	291	3077	550	1983

4 Estimate, by rounding to the nearest 10.

EXAMPLE	ROUND NOS.	ESTIMATE	EXACT ANSWER
103 + 68			
75 + 18			
180 – 21 – 56			
26 × 3			

5 Estimate, by rounding to the nearest 100.

EXAMPLE	ROUND NOS.	ESTIMATE	EXACT ANSWER
593 + 8763			
5862 –2097			
479 × 234			
7752 + 170			

6 Estimate, by rounding to whole numbers that you find easy to work with.

EXAMPLE	ROUND NOS.	ESTIMATE	EXACT ANSWER
2.7 + 7.9			
685 – 192			
860 × 65			
31.16 + 8.2			

7 Use your calculator to find the exact answers and compare them to the estimates.

Fold under

ANSWERS

1	30	40	510	7390	4310
2	400	300	3100	600	2000
3	7	13	1	20	50
4	170	4	100	300	
5	9400	3800	100000	39	
6	Answers will vary.				

Teaching Vocational Mathematics, Thiering, J, McLeod, J and Hatherly, S, NCVER 1992.

Worksheet 6

ESTIMATION EXERCISES 2

1 Estimate and calculate:

Step 1 Write easy round numbers for each example.
Step 2 Use these numbers to find estimates.
Step 3 Calculate the exact answers.
Step 4 Compare estimates and exact answers.

EXAMPLES	EASY NOS.	ESTIMATE	EXACT ANSWER
a $49.93 + 257.68 + 81.76$			
b $18.2 - 7.92 - 0.4$			
c $\frac{49.4 \times 24.5}{6.81 \times 4.7}$			
d $20.976 \div 3.71$			

2 How would you estimate the following?

a 0.49×7.18

b $0.4088 \div 5.6$

c $6.912 \div 0.15$

d $2.7^2 + 1.9^2$

e $\sqrt{156}$

f 7½% of $195.70

3 Would it have been better to do some of the items in question 2 twice on a calculator, and not bother with estimating? Which items?

Teaching Vocational Mathematics, Thiering, J, McLeod, J and Hatherly, S, NCVER 1992.

Worksheet 7

METRIC RELATIONSHIPS

THE 10 MILLIMETRE (1 CENTIMETRE) CUBE

How many 10mm cubes are there in a 100mm cube? .....

Fraction: Each 10mm cube is ..... of a 100mm cube.

Decimal: Each 10mm cube is ..... of a 100mm cube.

Filled with water, the 10mm cube contains 1 millilitre (mL) and the water would have a mass of 1 gram (g) approximately.

THE 100 MILLIMETRE (10 CENTIMETRE) CUBE

How many 100mm cubes are there in a cubic metre ? .....

Fraction: Each 100mm cube is ..... of a cubic metre.

Decimal: Each 100mm cube is ..... of a cubic metre.

Filled with water, the 100mm cube contains 1 litre (L) and the water would have a mass of 1 kilogram (kg) approximately.

THE 1 METRE CUBE (CUBIC METRE)

Filled with water, it contains 1000 litres or 1 kilolitre (1kL) and would have a mass of 1000 kilograms or 1 tonne (t) approximately.

One litre of water poured inside it would come ..... mm up the side of the cube.

Fold under

ANSWERS

The 10mm cube: 1000, $\frac{1}{1000}$, 0.001

The 100mm cube: 1000, $\frac{1}{1000}$, 0.001

The cubic metre: 1mm

Teaching Vocational Mathematics, Thiering, J, McLeod, J and Hatherly, S, NCVER 1992.

4 : Formulas

Many teachers comment that their students cannot do transposition of formulas. There are several reasons why this is such a difficult area for students, including:

- terminology
- the place of formulas in school mathematics

Terminology

The words 'transposition' and 'transpose' are not often used in maths teaching in the years before students enter vocational courses. The different words being used are interesting:

- 'transposition' has become 'change of subject'
- 'transpose for y' has become 'change the subject to y'

This change is seen as an improvement in schools because maths is taught as a language. Before using formulas students work with sentences.

For example:
5 + □ = 17 and
3a + 7 = 19

So a formula like the area of a trapezium is seen as a sentence:

$A = \frac{1}{2}(a + b)h$
'A' is the subject and '=' is the verb

If it is changed to:

$$h = \frac{2A}{a+b}$$

then 'h' has become the subject of the sentence. This links up with other language experiences. Compare these two sentences where the subject has been changed.

That tall blond boy is the fastest runner.

The fastest runner is that tall blond boy.

The emphasis on 'changing the subject' reinforces the way that maths is taught for understanding, rather than by rules. Transposition means 'crossing over', but there is a lot more to changing the subject of formulas than just passing letters from one side to the other in a kind of special routine. The old rule, 'change sides, change signs', is inadequate to describe all the processes, and it can also be misleading. If students follow it slavishly they make the following type of error with the plus and minus signs:

$\frac{1}{2}P = a + b$

$\therefore\ a - b = \frac{1}{2}P$

If you try to set out rules for the steps to be followed in changing the subject of formulas it becomes very complicated and even a slightly different formula has to have a new method. The best thing is to teach methods based on an understanding of what is going on. Basically, at each step students learn to do the same mathematical process to both sides of a formula.

Doing the same mathematical process to both sides of a formula

These processes include:

- adding (or subtracting) the same number
- multiplying (or dividing) by the same number
- taking square roots of both sides
- squaring both sides

Many teachers like to use an actual beam balance, or sketches of a balance, to reinforce the idea of treating both sides equally. It is also useful to write the two sides of a formula on two separate pages of scrap paper, making sure that any change made to one page is also made to the other page.

Another valuable approach is backtracking from a formula to see how it was originally developed. This makes it easier to change the subject.

Place of formulas in school mathematics

Formulas are only a small part of school mathematics. They are used as a tool for finding a value that is needed for a practical task.

For example:

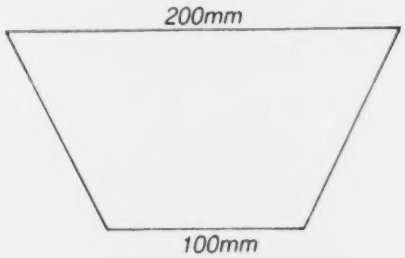
- finding the interest to pay on a loan
- calculating the area of a plot of land
- finding how many litres a tank holds

Most school students do not do any other work with formulas.

Change of subject of formulas is usually taught by a substitution method which gets rid of all the letters in the formula except the one that is needed. Students find it easier to work with numbers because they are less abstract than letters.

For example:

Find the height of the trapezium in this diagram. The area is 12000mm^2



Numerical method

$$A = \tfrac{1}{2}(a + b)h$$
$$12000 = \tfrac{1}{2}(100 + 200)h$$
$$12000 = \tfrac{1}{2} \times 300\ h$$
$$12000 = 150\ h$$

Divide both sides by 150

$$\frac{12000}{150} = \frac{150\ h}{150}$$

$$80 = h$$

$$h = 80$$

The height is 80mm

Traditional method

Change of subject of formulas in which letters are moved about before the numbers are put in is only taught to a small percentage of school students. Here is the way the above example would be done.

$$A = \tfrac{1}{2}(a + b)h$$

Multiply both sides by 2

$$2A = (a + b)h$$

Divide both sides by (a + b)

$$\frac{2A}{a + b} = \frac{(a + b)h}{a + b}$$

$$\frac{2A}{a + b} = h$$

$$h = \frac{2A}{a + b}$$

$$= \frac{2 \times 12000}{100 + 200}$$

$$= \frac{24000}{300}$$

$$= 80$$

If students want to check back on their work they can read the lines:

- 'multiply both sides by 2' and
- 'divide both sides by (a + b)'.

These are like signposts showing the track which was taken. As the students have more practice the words are omitted and the example looks like this:

$$A = \tfrac{1}{2}(a + b)h$$

$$2A = (a + b)h$$

$$\frac{2A}{a + b} = \frac{(a + b)h}{a + b}$$

$$h = \frac{2A}{a + b}$$

Rules have been replaced by a process that makes sense. All the steps are based on *doing the same to both sides of an equation* so that the verb, 'is equal to', continues to be true. Some people have succeeded in the past by using rules, but rules will only work if the student is keeping up constant practice. The dangers are that rules do not transfer easily to new situations and that a half-remembered rule is of no value.

Teaching formulas in vocational courses

Here is a typical method for changing the subject of a formula taken from a plumbing textbook. The last step would confuse many students.

$$grade = \frac{rise}{distance}$$

i.e. $$G = \frac{R}{D}$$

$\therefore$ by transposition,

$$R = G \times D$$

An alternative method is recommended because it follows on well from school methods.

$$grade = \frac{rise}{distance}$$

$$G = \frac{R}{D}$$

Multiply both sides by D

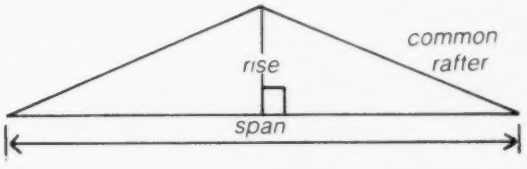
$$G \times D = \frac{R}{\not{D}_1} \times \frac{\not{D}^1}{1}$$

$$G \times D = R$$

$$\therefore R = G \times D$$

Another formula which is common to many courses is Pythagoras' theorem.

For example:
The span of a roof is 10 metres with a rise of 2.5 metres. Find the length of the common rafter.



The usual teacher's way to do this question in a plumbing or building course is:

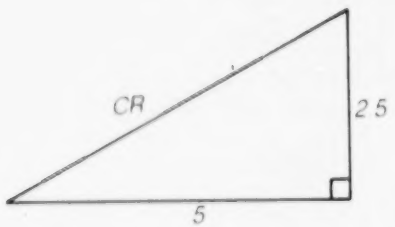
$$CR = \sqrt{\tfrac{1}{2}\,span^2 + rise^2}$$
$$= \sqrt{5^2 + 2.5^2}$$
$$= \sqrt{25 + 6.25}$$
$$= \sqrt{31.25}$$
$$= 5.6$$

NOTES

Students may have difficulty because:

1. The formula does not look like Pythagoras' theorem as they were taught it at school, $c^2 = a^2 + b^2$
2. In algebra, CR means C × R and $\tfrac{1}{2}\,span^2 = \tfrac{1}{2} \times (span)^2$. It should be written as $(\tfrac{1}{2}\,span)^2$.
3. The square root is over a very complicated set of numbers.

The method which would be used by students following their school mathematics is:



$$CR^2 = 5^2 + 2.5^2$$
$$= 25 + 6.25$$
$$= 31.25$$
$$\therefore CR = \sqrt{31.25}$$
$$= 5.6$$

NOTES

1. A new triangle is drawn to show the needed lengths.
2. The equation is easily recognised as Pythagoras' theorem.
3. The square root is used over one number only.

Summary

- The words 'transposition' and 'transpose' are new words to the majority of students.
- Most students have never worked with formulas in which the letters are moved around. The numerical method is much more common.
- Change of subject should be explained by the method of doing the same process to both sides of an equation and not by rules.
- Lessons with formulas should build on established school methods wherever possible.

5 : Decimals

Decimals have taken over in importance from fractions for recording parts of a whole. This change results from decimalisation of money, metrication of measurements and the almost universal use of calculators and computers. Even if students always use a calculator, they still have to understand the meaning of decimal notation, different ways of using zeros, order of size in decimals and be able to round to a given number of decimal places. In order to be able to estimate, they also need to be able to multiply and divide by multiples of 10, and understand what happens when you multiply and divide by numbers less than one.

Meaning of decimal notation

Students need a strong mental link between the number of decimal places and the number of equal parts, preferably without using fraction notation. Real-life examples will help them to understand.

- One decimal place is used to show 10 parts: petrol prices, timber lengths, trip meter distances.
- Two decimal places are used to show 100 parts: supermarket-shelf price tags, race record times, e.g. 59.48 s or 1:07.38.
- Three decimal places are used to show 1000 parts: pre-packaged meats in supermarkets, steel lengths, e.g. 2.700m.

Ways of teaching the concept of decimals and the meaning of decimal notation include:

- discussion of recording instruments such as trip meters, cash registers and electronic scales, preferably with photographs or models;
- concrete aids, e.g. coins, base-10 blocks (described in chapter 3). If you give the 100mm cube the value of 1, the blocks then give you pieces to show the relative size of 0.1 (the 100mm × 100mm squares), 0.01 (the 100mm × 10mm rods) and 0.001 (the 10mm cubes). Using these pieces, students can construct numbers of different value and see, for example, what happens when you add 0.001 to 0.009. They can see clearly the different sizes of numbers, e.g. how much smaller 0.009 is than 0.01, which will help them work out order of size in decimals;
- comparison of the decimal system with other base-10 numbering: room numbers in high-rise buildings, car number plates, phone numbers.
- comparison of decimals to other systems not based on 10, e.g. parts of an over in cricket (5.2 = 5 overs 2 balls), ages of children (1.3 = 1 y 3 mo).

Use of zero in decimals

Zero is used in a variety of ways in decimal work.

- Zero as a place holder, e.g. $16.08.
- Zero as a type of punctuation, e.g. 0.75.
- Zero to show degree of accuracy, e.g. 1.798 = 1.80 (correct to 2 decimal places).
- Zero in equivalent decimals, e.g. 1.6 on a calculator means 1.60 if you are working with money.
- Filling in zeros:
  When using pen and paper methods for adding decimals with 'ragged ends', students should always fill in the missing zeros.

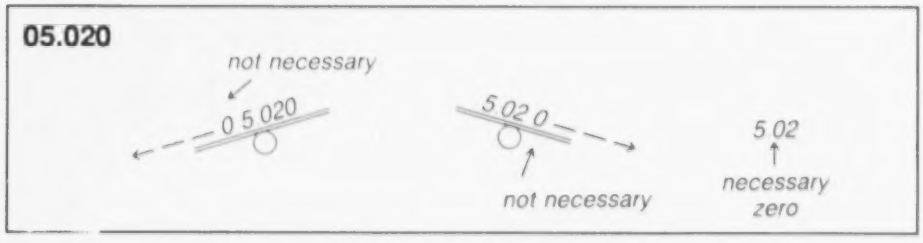
  0.36 + 1.9 becomes 0.36
  +1.90

- Zeros in metric measuring:
  It is helpful when recording lengths in metric to fill in three decimal places using zeros, e.g. 1.8m lengths of steel may be recorded as 1.800m, which gives a quick conversion to 1800mm.

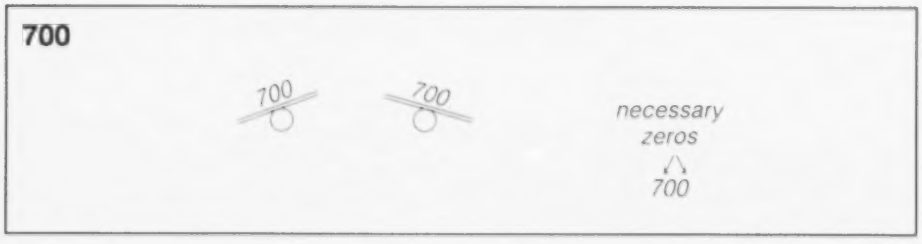
Essential zeros

Essential zeros and non-essential zeros can be sorted out by a see-saw method. When a number is tilted a zero will roll off if it is not an essential zero[1].

Note: A decimal point or another digit will stop zeros rolling off.



The idea of rolling zeros works for whole numbers also if you stress that all whole numbers have an invisible decimal point straight after the units digit.



[1]Adapted from 'A Case for Rolling Zeros' by C. W. Bell, in *Arithmetic Teacher*, Vol. 28, No 3, Nov. 1980

Make sure students realise that unnecessary zeros can be left out when entering numbers into a calculator. For example, for 1.20, enter 1.2; for $27.00, enter 27 without affecting the accuracy.

Order of size in decimals

A way to test whether or not students have understood the concept of decimals and the use of zeros is to ask them to place several decimals in order from smallest to largest. This is a common criterion reference test item.

For example:
Arrange from smallest to largest:
0.9, 0.096, 0.906, 0.96

Decimal Order Cards are a good teaching aid for order of size in decimals. If students cannot order the cards correctly it shows that some decimal concepts still need development.

A set of 24 cards of playing-card size are printed on one side with one-, two- and three-digit decimals, 0.008 to 0.96. Students may work alone or in small groups. The cards are shuffled and drawn one at a time from the pack and placed in a row from smallest to largest. Without the teacher being involved the pack is played out and, when the cards are turned over, the correctness of the order chosen is revealed by small numbers, 1 to 24. If there are errors they have to be corrected with a minimum of teacher intervention. Some people put the smallest on the right, contrary to the conventions for the number-line and measuring, and this has to be altered. Self-correction brings out many misconceptions on decimal notation. Most students eventually decide on a method of mentally filling in zeros to three places; some use a system that is similar to alphabetical order.

Decimal Order Cards are available from the Adult Literacy Information Office, TAFE Commission of NSW.

Rounding

Some students ask, 'Do decimals go on past three places?'

Students will come across decimals with four or more places in a variety of calculations: multiplying, dividing, squaring and taking square roots. They must get used to seeing strings of digits on their calculators. They will ask, 'What do I do with all these numbers?' Handling this requires rounding (or approximation).

Rounding to 1, 2 or 3 decimal places is an essential skill. Many students do not realise that answers in maths are often not exact; answers are as accurate as you want them to be for the practical problem you are solving.

A strategy for rounding decimals:

Look at the part of the number that is not needed. If there is half the unit of measure or more, take the answer up. If there is less than half, take the answer down.

Do some calculations with a non-decimal system such as time.

For example:
Write to the nearest hour, 4 hours 38 minutes.
Write to the nearest week, 3 weeks 2 days.

When students understand the concept, change to decimals. The decimal system has 10 digits; 5 small and 5 large.

Small: 0 1 2 3 4	Large: 5 6 7 8 9
Small digits, make no change	Large digits, go up one.

When rounding, give students something to do with their pen besides just sitting there and looking at the numbers! Draw a cut-off mark after the number of decimal places needed.

For example:
Write 2.786 correct to 2 decimal places.
The cut-off mark goes after the 8: 2.78/6
Check the digit on the right of the cut-off mark to see if it is large or small.
If large, go up one.

+1
2.78/6 Answer = 2.79

Use a variety of examples and include complicated ones with zeros. These will again remind students that 1.8 = 1.80 = 1.800, except when they show the degree of accuracy.

For example:

1.796	to	2 decimal places	= 1.80
4.996	to	2 decimal places	= 5.00
2.03	to	1 decimal place	= 2.0
7.999	to	1 decimal place	= 8.0

This sort of practice should also be given in problems. For example, when preparing money problems, choose numbers that will give calculator answers to only one, or more than two, decimal places so that students get real practice in rounding.

For example:

6 items at $3.80	=	23.1
	=	$23.10
$58.85 split between 3	=	19.16666667
	=	$19.17
85.6 litres of petrol at 68.8 cents per litre	=	5889.28 cents
	=	$58.89

Multiplying and dividing by 10, 100, 1000

To do this we teach students to move the point. However, there is a major source of confusion here. Is the decimal point moving, or are the digits moving?

For example:
Multiply 7.25 by 10, 100 and 1000.
10 x 7.25 = 72.5
100 x 7.25 = 725
1000 x 7.25 = 7250

Obviously, the 7, 2 and 5 are moving, *not* the point.

However it is still better to say, 'Move the decimal point'. Reasons for sticking with this old rule are:

- because students have always done it this way at school;
- because it is neat and tidy;
- because we don't usually work with ruled columns.

Bookkeepers and sales assistants work in ruled columns and they move the numbers, not the point, to avoid mistakes.

For example:
Find the cost of a book @ $16.80 if 10% discount is allowed.

Cost	*$16*	*80*
less 10%	*1*	*68*
Balance	*$15*	*12*

Students who have trouble with keeping decimals under each other could turn their paper so that the lines become columns.

A source of error when students are 'moving' the decimal point, is that they are not sure what they are counting, the digits or the spaces, and so they often count some of both. These students need to be encouraged to count with the tip of their pen placed clearly between the numbers.

One common application of multiplying and dividing decimals by multiples of 10 is metric unit conversions, something which many students find difficult. Worksheet 5.1 is designed to help students see the patterns involved in metric conversions. It is essential to make sure that students have clear concepts of size before giving them an exercise such as this worksheet (see chapter 3).

Multiplication and division by numbers smaller than one

Students concepts of multiplication were first formed in primary school and their ideas may not have grown significantly since then. This means they feel quite sure

about two results: **a** if you multiply two numbers together the answer will always be bigger than either of the numbers you started with, e.g. $7 \times 8 = 56$ and **b** if you divide one number by a second number, the answer will always be smaller than the number you started with, e.g.

For example:
$12 + 3 = 4$

The trouble is that this is only true if both the numbers being multiplied are greater than one. As soon as students work with decimals smaller than one they find that the false rule on which they have relied for so long breaks down. Practical calculations use a lot of numbers smaller than one, especially since metrication, and it may be that the vocational teacher is the person who has to help the students to develop their concepts to cope with confusing results like:

$7 \times 0.8 = 5.6$ and
$12 + 0.3 = 40$

The facts are that:

- If you multiply any number at all by a number less than one, your answer will be smaller than the number you started with.
- If you divide any number at all by a number less than one, your answer will be greater than the number you started with.

Some practical suggestions for dealing with this difficulty are:

- Use well-known materials like money, timber, pipes and fabric with hands-on exercises: 7×0.08 is the same as 'find the cost of 7 screws at 8 c each; $12 \div 0.3$ is the same as 'find how many 0.3 m lengths of wood (or pipe) can be cut from a 12 metre length'. The lengths can be marked out with chalk to show how sensible the answer is.
- Use a calculator to go through a sequence like:

  $7 \times 8 = 56$
  $7 \times 0.8 = 5.6$
  $7 \times 0.08 = 0.56$

  and in division show that the answers grow larger as the divisor gets smaller.

  $12 + 3 = 4$
  $12 + 0.3 = 40$
  $12 + 0.03 = 400$ and so on

- Talk through what the mathematical 'sentence' means.

  For example:
  7×8 means I want 7 lots of 8, *more* than I started with. But 0.7×8 means I don't even want one lot of 8, I only want 0.7 of 8 i.e. *less* than I started with.

  $12 + 3$ can mean 'How many 3s in 12?' The answer is not very many, only 4.
  $12 + 0.3$ also can mean 'How many 0.3s in 12, i.e. how many small things less than 1 can I get out of 12 (or 12 lots of 1)?' The answer has to be a large number, many more than 12.

- Find practical examples specific to the vocational area and work through them with real materials. Encourage students to talk about what is happening, in their own words. Then write down the processes in symbols and calculate the answers. Discuss how the results match the real situation.

Worksheet 8

METRIC CONVERSIONS

Do these metric conversions without using a calculator.

1	1246g	to kg		2.382kg	to g
	7380mm	to m		16.2m	to mm
	27 980mL	to L		3.23L	to mL
2	268g	to kg		0.267kg	to g
	450mL	to L		0.385m	to mm
	267mm	to m		0.996L	to mL
3	45mm	to m		0.012m	to mm
	66g	to kg		1.099L	to mL
	20mL	to L		0.038kg	to g
4	4g	to kg		0.006m	to mm
	9mm	to m		0.002L	to mL
	8mL	to L		8.001kg	to g
5	237ml	to L		1.2L	to mL
	0.029kg	to g		1547.0g	to kg
	3mL	to L		45.06m	to mm
	0.004L	to mL		78.200g	to kg
	59mm	to m		9.1m	to mm

Fold here

ANSWERS

1	1.246kg	2382.0g	2	0.286kg	267.0g
	7.380m	16 200.0mm		0.450L	385.0mm
	27.98L	3230.0mL		0.267m	996.0mL
3	0.045m	12.0mL	4	0.004kg	6.0mm
	0.066kg	1099.0mL		0.009m	2.0mL
	0.02L	38.0g		0.008L	8001.0g
5	0.237L	1200.0mL			
	29.0g	1.547kg			
	0.003L	45 060.0mm			
	4.0mL	78.2kg			
	0.059m	9100.0mm			

Teaching Vocational Mathematics, Thiering, J, McLeod, J and Hatherly, S, NCVER 1992.

6 : Fractions

These days calculations are done with decimals, but concepts of a fraction and the skill of handling equivalent fractions are still important.

Common fraction ideas

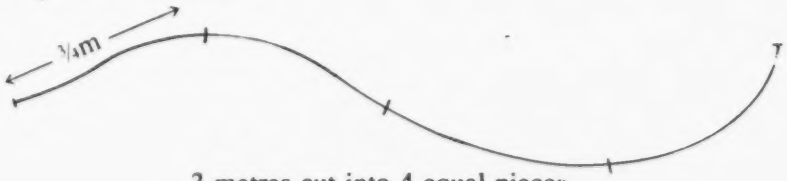
Many adults never use more complicated fractions than ½ and ¼. These occur in telling the time and in sharing things equally, such as food portions. The best shape to use to explain fractions is a circle as it can be cut into any number of equal parts by drawing radii and one or two missing parts show up immediately as the shape is clearly incomplete. This does not happen when using squares on rectangles; however, the most handy teaching aid for quick explanations of difficulties is a piece of paper, folded to give halves, quarters, eighths and so on. It is wise to check that a student who is having trouble can explain the concept of a fraction or ratio with a concrete object like a piece of paper because these concepts are usually not well formed until the middle teen years. Many young adult students may have incomplete fraction concepts.

Two meanings for the same fraction

Fractions grow out of division, because wholes are divided into equal parts. However, there is another way to look at a fraction. This is linked to the concept of sharing, where several wholes are shared in equal lots.

Consider the fraction $\frac{3}{4}$ In almost all school textbooks and in many vocational classrooms you would find a sketch being used to explain this fraction. Usually there would be a circle cut into four pieces with three of them shaded.

This is one meaning of three-quarters. It is also necessary to give an equal amount of attention to the other meaning, which can be shown by taking a piece of string three metres long and folding it in half and in half again to give four pieces. If you cut them, each one is a quarter of the 3m you started with and each when measured, is 750mm or $\frac{3}{4}$ of a metre long.



3 metres cut into 4 equal pieces

So $\frac{3}{4}$ can mean one whole, cut into four equal parts, with three put together to give you three quarters; it can also mean three wholes being shared or divided into four equal lots, each lot being a quarter of three.

$$\frac{3}{4} = 3 \times \frac{1}{4} \text{ or } \frac{3}{4} = \frac{1}{4} \times 3$$

For example:

$\frac{3}{4}$ of 1 hour is 45 minutes, and

$\frac{1}{4}$ of 3 hours is also 45 minutes, so $\frac{1}{4}$ of 3 = $\frac{3}{4}$ of 1

The reason for emphasising both of these meanings of a fraction is that the second way of looking at fractions is the better one for teaching students that the division operation is being done by the 4; that the meaning of $\frac{3}{4}$ is 3 divided by 4, not 4 divided by 3. Understanding this will enable them to change fractions to decimals using a calculator by pressing the right buttons.

When mathematics tests start to take notice of the widespread use of calculators in industry, schools, shops and homes there should be changes in the type of items written to test knowledge of fractions and decimals. Such a question might be:

Which of these is the correct way to change

$\frac{3}{4}$ to a decimal: $3 \div 4$, or $4 \div 3$?

It should not be assumed that vocational students will have an awareness of both these meanings for a fraction. Teachers know that many students are not able to change fractions to decimals using the correct calculator buttons, and that they seem unable to check the reasonableness of the decimal equivalent which they have found. How can $\frac{3}{4}$ be equal to 1.3333?

Equivalent fractions

A chart of multiples is useful to explain equivalent fractions. Displayed on an overhead projector, or handed out on separate pages to each student, the chart looks like a full set of multiplication tables facts.

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

This is not just the 'tables', but a simple teaching aid for equivalent fractions. If you look at the top two lines not as two rows of numbers but as one row of fractions you will see a set of equivalents for $\frac{1}{2}$

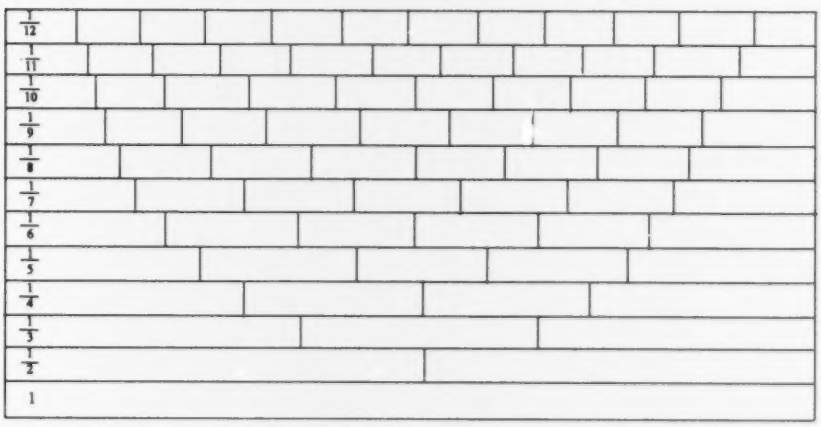
$$\frac{1}{2} \quad \frac{2}{4} \quad \frac{3}{6} \quad \frac{4}{8} \quad \frac{5}{10}$$

Any two lines give a set of equivalent fractions. You could start with $\frac{5}{6}$ or $\frac{8}{9}$ for example. If the chart is folded, you can show the equivalent fractions for more fractions such as $\frac{3}{10}$ or $\frac{4}{7}$. Use of the chart reinforces the fact that all the equivalent fractions have come from the multiplication tables. Students can find even more fractions by extending the pattern of multiples past 10 times to 11, 12, and even 15 if they are likely to meet fractions of this size in their work.

You could encourage students to change all the fractions in a pair of lines to decimals, thereby seeing for themselves that they really are all the same number.

Another good way of helping students understand equivalent fractions is by folding a piece of paper in half, and in half again and so on. Students can count and name each successively smaller rectangle as $\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{8}$ $\frac{1}{16}$ and so on. They can then see that the same size piece can be labelled $\frac{1}{2}$ $\frac{2}{4}$ $\frac{4}{8}$ $\frac{8}{16}$ and so on.

A fraction wall as shown below can also be used as a model of equivalent fractions. Have students move a straight edge, such as a pen or a ruler across the page and note where the vertical lines match up. They can then find a variety of equivalent fractions. They can also see how $\frac{3}{4}$ could never be expressed as thirds, or how $\frac{1}{3}$ can never be expressed as tenths. This is a good lead up to explaining recurring decimals.



Simplifying fractions

Simplifying fractions becomes a process of finding any smaller number which will divide evenly into both the numbers of the fraction. It is not necessary to find the biggest such number immediately. When simplifying a fraction like $\frac{42}{56}$ it is not wrong to divide both numbers by 2 first and then by 7, even though in the old days we were expected to find the highest common factor (HCF) first and then divide by 14 to get $\frac{3}{4}$ in one step. The word 'factor' for the numbers used to divide both numerator and denominator should not cause problems as it is common terminology from primary and secondary school.

(See chapter 2 for discussion about developing the concept of factors.)

Encourage students to think about the particular numbers involved before testing for common factors, e.g. if both are even numbers they will be divisible by 2, if they both end in 5 or 0, they will be divisible by 5 and so on.

The chart of multiples also gives students a starting point for finding suitable factors when they have to simplify fractions. With $\frac{42}{56}$ they can find 42 and 56 in the column that is headed by 7, so division by 7 is right. The resulting fraction $\frac{6}{8}$ obviously can be simplified to $\frac{3}{4}$. Many students will want to write out what they are doing as follows, rather than using 'cancelling'.

$$\frac{42}{56} = \frac{42 \div 7}{56 \div 7} = \frac{6}{8} = \frac{6 \div 2}{8 \div 2} = \frac{3}{4}$$

Clearly these students show that they know what they are doing. In time they will probably speed up. It is better to wait until they realise for themselves that the middle steps are not necessary, rather than the teacher giving them short cuts. When they shorten the work themselves it is an indication that they understand.

For many students the chart of multiples is an eye-opener to the patterns on which the equivalent fractions are based. They should respond positively to something which makes sense of processes which may in the past have been very hit and miss.

7 : Percentages

Percentages are used in every vocational area. When costing, pricing or wastage are involved, calculations with percentages are certain to be involved. It is therefore important that students understand and are able to operate effectively with percentages.

Common percentage ideas

Giving students some idea of the purpose and usefulness of percentages should help their understanding. One aspect that may interest them is the fact that percentages allow comparisons to be made easily. An example that most students can relate to is comparing marks on two tests marked out of different amounts e.g. 32 out of 40 and 29 out of 35. At face value, it would appear that 32 is the higher mark, but converting both to percentages shows that 32 was 80% and 29 was nearly 83%. You can introduce the idea here that converting to a percentage changes both marks to a mark out of 100 so that they can be compared.

You can also link percentages to other common objects, like thinking of percentages in terms of cents in the dollar. With or without the actual coins on the table you can establish that one cent is one per cent of one dollar and extend this to any other percentage such as 35 cents is 35% of $1. This approach gives clear meaning to the concept, using a real-life system, rather than a specially designed mathematical aid.

Another interesting aspect of percentages is their power to summarise what would otherwise be a long list of numbers. For example, imagine a shop advertising a sale that listed the actual amounts by which each item in stock had been reduced. You might be looking at a list of numbers ranging from 65 cents to $65. Instead, shops advertise a 10%-off sale and potential customers immediately have an ideas of the extent of the savings they will make, and realise that if the original price is small, it won't amount to a lot, but if the original price is large, the savings could be substantial.

A percentage is a common scale that most people can interpret and relate to easily. In order to be able to use this common scale effectively, students need to be able to convert between common fractions and percentages. At a minimum they need to be familiar with the following:

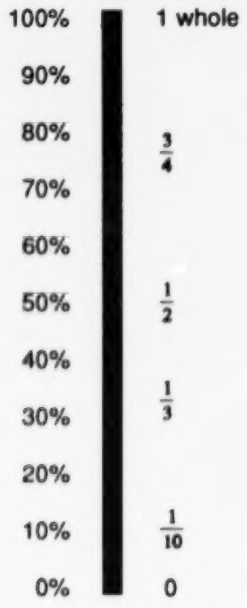
$50\% = \frac{1}{2}$

$33.33\% = \frac{1}{3}$ and $66.67\% = \frac{2}{3}$

$25\% = \frac{1}{4}$ and $75\% = \frac{3}{4}$

$20\% = \frac{2}{10}$ and $10\% = \frac{1}{10}$

A visual way of presenting the concept of percentages is a simple vertical line:



Students can see that 50% is half-way, 33% a third of the way and so on. Encourage students to attach word meanings to different percentages, e.g. 99% means almost all, 1% is hardly any, 48% is almost a half and 35% a bit more than a third.

Looking at the line above, students should come to realise that you can move from left to right and back because the values are equal and interchangeable.

For example:

$\frac{1}{10}$ of one whole can also be thought of as:

$\frac{1}{10}$ of 100% or $\frac{1}{10} \times 100\%$ which matches 10% and

$\frac{1}{3}$ of one whole = $\frac{1}{3} \times 100\%$ which matches $33\frac{1}{3}\%$.

Without a % button, the calculator process is

[1] [+] [3] [×] [100] [=] which gives 33.333 . . . without a % sign.

With a % button, the process is quicker:

[1] [+] [3] [%] which gives the same answer.

Have students do these calculations so they can see the precise answers matching the position on the vertical line.

Comparing two quantities as a percentage

You can then use the vertical line to develop the concept of comparing any two quantities. If you want to compare them as a percentage, think of them first as a fraction.

$10 off the price of something costing $113 is $10 out of the whole, $113.

Draw a line similar to the one above with $113 at the top on the right instead of 1 whole, and $0 at the bottom. Mark in roughly where 10 comes on the scale 0 to 113. It is near 10%.

Remind students that 'out of' means 'divide' i.e. $10 \div 113$ or $\frac{10}{113}$

Students can then work out the percentage exactly:

$\frac{10}{113}$ of $100\% = \frac{10}{113} \times 100\% = 8.8\%$.

The same method applies with decimals. To find the percentage wastage when there is 0.75 kg waste on a leg of lamb weighing 2.6 kg, you mark the right-hand scale from 0 to 2.6 and estimate where 0.75 will be. It is near 25%.

The exact answer is $0.75 \div 2.6 \times 100 = 28.8\%$.

Students will soon see that the way to convert to percentages is to divide the part by the whole and multiply by 100. Note that this process works even if the amount you are converting to a percentage is greater than the whole. You just end up with a percentage greater than 100%.

If students are operating with fractions constantly, it is a good idea to suggest that they buy a calculator with a percentage button. You can show them that the % button will save them pressing so many keys. The percentage button saves pressing these buttons:

Decimals and percentages

Once you have established the method for comparing quantities as percentages, it is an easy step to show the link between decimals and percentages. By establishing that it is basically getting a fraction and multiplying it by 100, you can show students that they could alternatively find the decimal equivalent of a fraction and then multiply that decimal by 100. Lay out a few common examples.

FRACTION	PROCESS	DECIMAL	PERCENTAGE
$\frac{1}{2}$	$1 \div 2$	0.5	50
$\frac{3}{4}$	$3 \div 4$	0.75	75
$\frac{4}{5}$	$4 \div 5$	0.8	80
$\frac{1}{20}$	$1 \div 20$	0.05	5

You can then include uncommon examples:

$\frac{3}{40}$	$3 \div 40$	0.075	7.5
$\frac{24}{60}$	$24 \div 60$	0.4	40
$\frac{82}{512}$	$82 \div 512$	0.16	16

Asking students why they think we multiply by 100 rather than just working with the decimal might generate some useful discussion. Lead them with examples like an advertisement claiming '0.1 off all stock'. Some issues to discuss are: we use percentages rather than decimals so we don't have to operate with numbers smaller than one; people find percentages are easier to say than naming decimals. Such discussions might help students see the commonsense side of maths.

Finding a percentage of a quantity

Most students can work out problems involving a percentage of a given quantity fairly easily using the percentage button on a calculator. The important skill for all students is to estimate their answer before using the calculator.

Estimating a percentage of a quantity can also be done effectively with the vertical line that we used before.

Step 1 Round the whole quantity to an easy number.

Step 2 Write this easy number opposite 100%.

Step 3 Write in the percentage at the appropriate place between 0 and 100%.

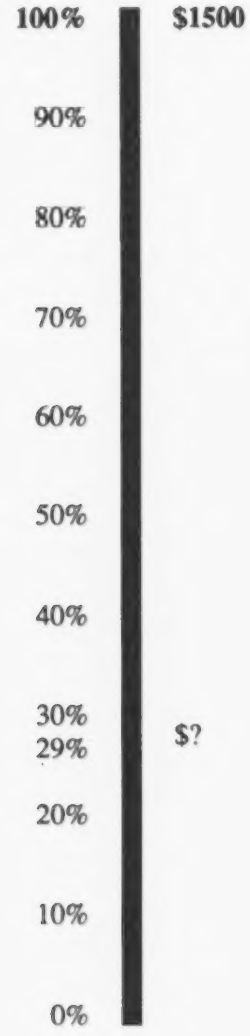
Step 4 The estimate will be the quantity that is opposite the percentage. You can find its value based on its distance between 0 and the whole quantity.

For example:

Find 29% of $1430

Step 1: Round $1430 to $1500.

Steps 2, 3:



Step 4: You can see that 29% is not quite one third of the way up. One third of $1500 is $500 so the answer should be a bit less than $500. *(The actual answer is $414.70.)*

Another useful approach to estimating is to break down the percentages into parts that are easy to work with. Make sure first that students are confident that:

$10\% = \frac{1}{10}$ and finding $\frac{1}{10}$ is the same as dividing by 10

$50\% = \frac{1}{2} \Rightarrow$ divide by 2

They can then use combinations such as:

$5\% = 10\% \div 2$ $\Rightarrow$ divide by 10 and then divide by 2

$15\% = 10\% + 5\%$ $\Rightarrow$ divide by 10, then divide that result by 2, and add your answers

$90\% = 100\% - 10\%$ $\Rightarrow$ divide by 10 and then subtract the result from the total amount

Discovering and applying these kinds of relationships can give students confidence in their own mathematical ability.

If your students need to be able to work out percentages without using a percentage button on a calculator, it is important to teach them a method that is both conceptually sound and makes sense.

If you need to find 30% of 24, one way of doing it is $30 \div 100 \times 24$, another way is $24 \times 30 \div 100$ and yet another way is $24 \div 100 \times 30$. All are mathematically correct, but the logic in the first two is hard to explain. The third method is preferable.

If you approach the calculation logically, Step 1 is to get 1%, i.e. divide by 100 and Step 2 is to get 30% i.e. multiply 1% by 30.

Therefore, it is far better to teach students $24 \div 100 \times 30$. Students will have a better chance of remembering the method because it makes sense. They can put into words what they are doing. e.g. '24 is 100% but I only want 30%. If I divide by 100, I'll know what 1% is and then I can multiply by 30 to get 30%'. This method of finding 1% first is called the *unitary method.*

Worksheet 9 has examples to use for investigating a variety of approaches to estimating a percentage of a quantity. It could also be the focus of group discussions, or students could write down their methods on the worksheet.

Working out 100% when given another percentage

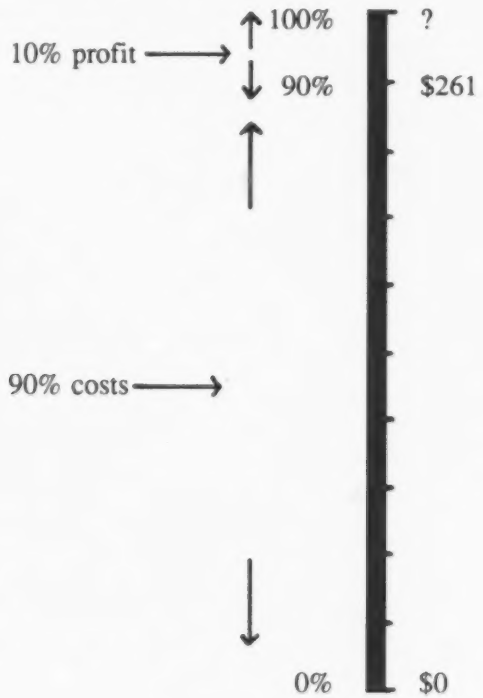
This is the kind of problem sometimes involved when costing a job. Usually costs and a percentage profit are given, and the price of the job has to be worked out. It is important to find out first whether the percentage profit in the particular work context is calculated as a percentage of the price of the job or as a percentage of costs.

Again the vertical line is a useful aid to students who are trying to work out what to do.

For example:
If the total costs of a job are $261, what will you charge the customer if you want a profit margin of 10% on the takings?

First put all the known information on the diagram. The selling price (or takings) is 100%.

Labelling parts of the line as costs and profit helps the student see that costs must equal 90%. Discussion will make it clear that we know 90%, which is $261 and we want to know 100%. From the diagram we can estimate the amount that should be opposite 100% is about $300.



The unitary method can be used in this problem too. If $261 = 90%, 261 ÷ 90 = 1%. This can then be multiplied by 100 to give 100% or the selling price.

$261 \div 90 \times 100 = 290$

If students then go back and write $290 level with 100% and compare it to the information already on the diagram they can check the reasonableness of their answer. $290 next to 100% looks like a sensible answer.

Worksheet 9

ESTIMATING PERCENTAGES

Find different ways to estimate the following percentages.

1 12½% of $94

2 5% of 15m

3 7½% of 47m^2

4 3½% of $9.30

5 15% of 480

6 85% of 1 586 tonnes

7 2½% of 121

8 90% of 6.5m^3

9 80% of 48kg

Teaching Vocational Mathematics, Thiering, J, McLeod, J and Hatherly, S, NCVER 1992.

8 : Worked examples

The following problems are taken from teaching materials for some popular vocational courses. On the right-hand page is what might be written on the board or in a student's book. On the left-hand page are notes on the problem solving process. It is not intended that these points would all be made by the teacher, but rather that they would come out through questioning and discussion with students. It is also expected that the technical explanations would be made at the same time as the mathematical explanations, using either models or the real thing.

The problems are each worked using a method chosen by the authors. Other methods are available and may be more commonly used in particular courses or at certain institutions. The methods chosen exemplify the four steps of the problem solving approach. The notes show the type of reasoning which would be talked through by a teacher who worked along the lines suggested in this book. Eventually such a teacher's students would also be using the thinking approach to tasks they meet in vocational contexts.

Worked example 1:
Notes on the problem-solving steps

GET TO KNOW THE PROBLEM

You have to make a taper that has a diameter 29mm at one end and 11mm at the other. The piece is 74mm long but 12mm of this is not tapered. Therefore, the tapered length will be 74 – 12 = 62mm. The diagram shows a cross-section of the piece.

The angle you have to work out is the angle formed by the side of the taper and a horizontal line. The best way to work out an angle is to find a right-angled triangle that includes the angle you want, so that you can then use trigonometry.

Looking at the diagram, imagine that the 11mm wide section on the right extends to the left through the middle of the taper. Now imagine that you remove this section, leaving one large triangle. Draw this simpler diagram.

You can see that this has not changed the slope of the sides. The height of this triangle will be the large diameter minus the small diameter, which you have just removed (29 – 11 = 18mm).

The angle formed where the two sides meet (called the included angle) is twice the angle you need to cut the taper. If you draw a line which cuts this angle in half, you will get two identical right-angled triangles. Each triangle contains the angle you want, which we will call 'a'. Add this to the diagram.

CHOOSE WHAT TO DO

If you look at either of the two right-angled triangles, you already know the length of the base (62mm). This is the side adjacent to the angle you want to know. You can also work out the opposite side it is 18 ÷ 2 = 9. So, now you know the opposite and adjacent sides to the angle you want to know. The trig ratio that uses the opposite and adjacent sides is the tan ratio.

WORK OUT THE ANSWER

Make an estimate first.

You change $\frac{9}{62}$ to easy numbers first and get an estimate of 0.1. You know 45° has a tan of 1, so an angle with a tan of 0.1 must be pretty small.

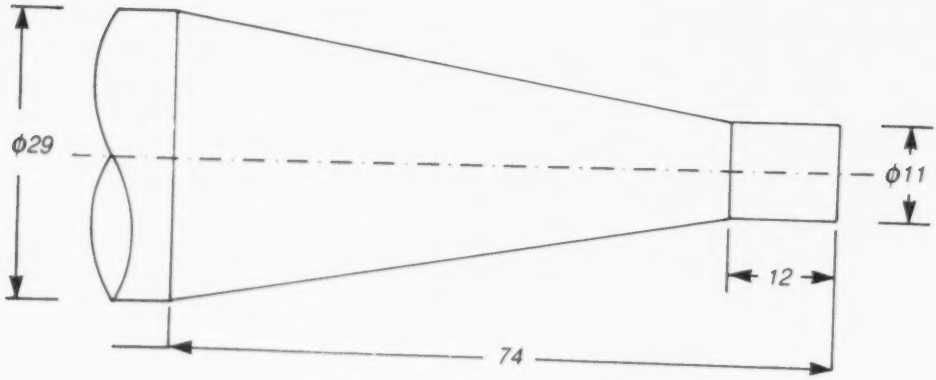
Then you calculate the angle using the fact that its tan ration is $\frac{9}{62}$.

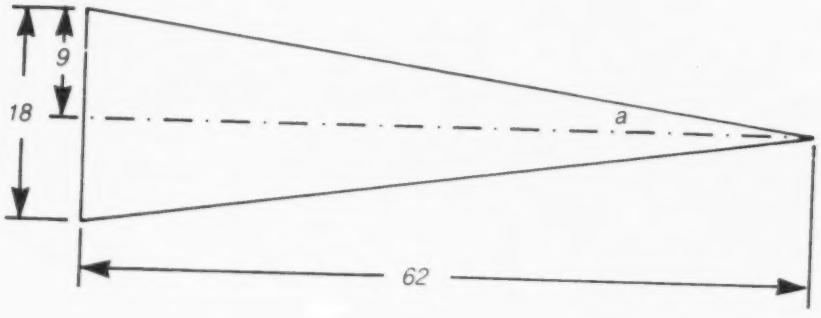
LOOK BACK

The answer is close to the estimate. Think about whether the answer seems reasonable. It is a small angle and it looks right for the diagram. It matches what you usually use in the workshop.

Worked example 1: Fitting and machining

You have to cut the taper shown in the drawing below. Work out what angle you would set the compound slide at.





$$\tan a = \frac{\text{Opposite}}{\text{Adjacent}}$$
$$= \frac{9}{62}$$
$$= 0.14516129$$
$$\therefore a = 8.26°$$
$$= 8°15'$$

Estimate:
$\frac{6}{60} = \frac{1}{10} = 0.1$

You would set the compound slide at 8°15' to cut the taper.

Worked example 2:
Notes on the problem-solving steps

GET TO KNOW THE PROBLEM
First, draw a plan of the roof. Remember the roof has two sides and a rise. The roofing sheets will be the length of the common rafter. The rise, half span and the common rafter form a right-angled triangle. The common rafter is the hypotenuse. Draw a second diagram to show this. If you work out the length of the common rafter, you will know the length of the roofing sheets. To make an order you will also have to work out how many sheets are needed. You would do it in two separate sections, like two problems.

a Length of roofing sheets

CHOOSE WHAT TO DO
As you know the length of two sides of a right angled triangle, you can use Pythagoras' theorem to work out the length of the third.

WORK OUT THE ANSWER
You can use your knowledge of the relative length of the hypotenuse and the longer of the other two sides of the triangle to get an estimate before calculating the answer. You will have to use the CR^2 formula and then get the square root.

LOOK BACK
This answer fits with your estimate. A roof with 3750 for ½ span would be likely to have roofing sheets just over 4 metres long.

b Number of sheets needed

CHOOSE WHAT TO DO
The effective cover is the width of one sheet, minus the overlap. You know the length of the ridge, so if you divide the length by 760, you will have the number of sheets needed for one side.

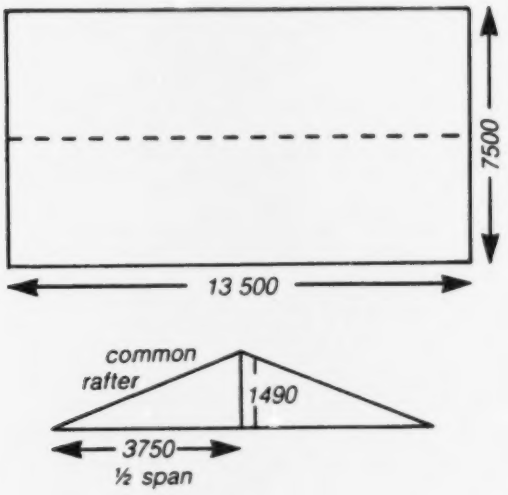
WORK OUT THE ANSWER
Estimate the answer before doing the division using rounded easy numbers. The exact answer is 17.76; however, you can't buy 0.76 of a sheet so you will have to buy 18 for each side. If the exact number had been 17.36 you may have been able to cut a sheet into two pieces. Then you would have bought 17 for each side plus one to split, 35 altogether.

LOOK BACK
Check the answer. The answer is close to the estimate. It sounds right, compared to what has been done at work lately.

Worked example 2: Plumbing or building

Calculate: (a) the length of roofing sheets
(b) the number of sheets needed to cover a gable roof

The dimensions of the roof are: ridge length 13 500, span 7500, rise 1490. Effective cover of the sheets is 760. (All measurements are in millimetres.)



(a) $CR^2 = rise^2 + (\frac{1}{2}\ span)^2$
$= 1490^2 + 3750^2$
$= 16\ 282\ 600$
$CR = \sqrt{16\ 282\ 600}$
$= 4035$

Estimate: The hypotenuse will be not much longer than 3750, probably around 4000. It can't be any more than about 5000.

The length of the common rafter and therefore the roofing sheets is 4035mm.

(b) No. of sheets for 1 side $= \frac{13\ 500}{760}$
$= 17.76$
$= 18$

Estimate: $\frac{14000}{700} = 20$
Total needed will be about 40

No. of sheets for 2 sides $= 18 \times 2$
$= 36$

The roof will take 36 sheets.

Worked example 3:
Notes on the problem-solving steps

GET TO KNOW THE PROBLEM

The idea of a brake is that a relatively small force (effort) on the end of the lever creates a much greater force (load) in the push rod that will eventually create a force large enough to stop the car. The pivot at the top acts as a fulcrum.

Draw a simpler diagram, showing the distances and the forces.

CHOOSE WHAT TO DO

We know two distances from the fulcrum and one force. It is a problem dealing with moments. The moments have to be equal.

WORK OUT THE ANSWER

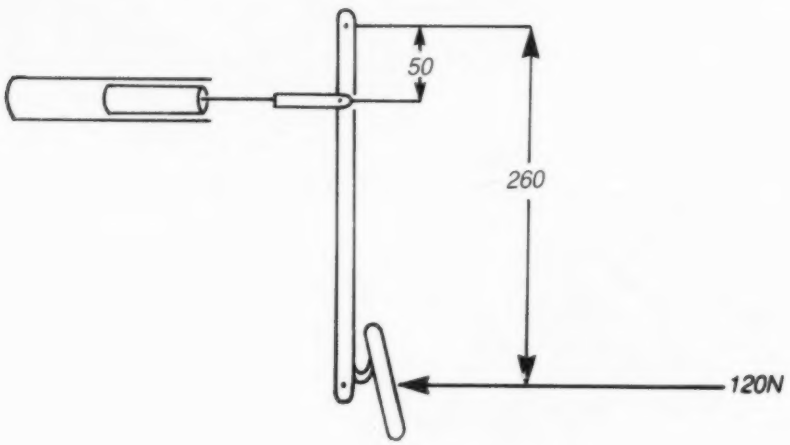
Write the moments equation and put the values in. Round to easy numbers and get a rough answer. Work out the load exactly.

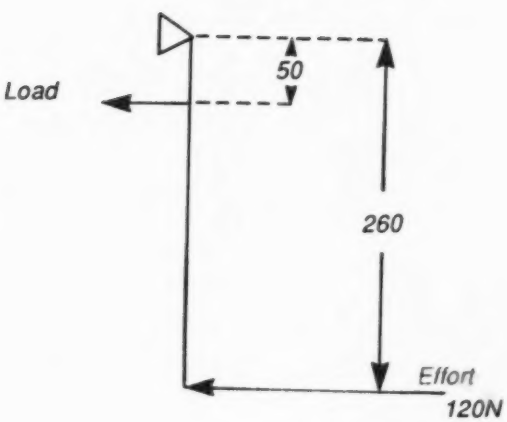
LOOK BACK

The answer is close to our estimate. The force exerted at the push rod is much greater than the original force. This makes sense because the brake is more powerful than the force of the driver's foot.

Worked example 3:
Automotive engineering

A 260mm long pendant type brake pedal is pivoted at the top end. The master cylinder push rod joins 50mm from the pivot. If the driver exerts a force of 120N on the pedal, what force will be exerted at the push rod?





$$\text{Load} \times 50 = \text{Effort} \times 260$$
$$= 120 \times 260$$
$$\text{Load} = \frac{120 \times 260}{50}$$
$$= 624$$

The force exerted at the push rod is 624N.

Estimate

$$\frac{120 \times \cancel{250}^{5}}{\cancel{50}} = 600$$

Worked example 4:
Notes on the problem-solving steps

GET TO KNOW THE PROBLEM

Establish at the beginning that the final gear will be slower than the driver because the driver turns many times to make the end gear turn once. In fact, the final gear will be much slower than it would be if there were only the driver and the final gear in a simple gear chain because the effect of the middle gears is to slow it down.

In this problem, one turn of the driver 1 will mean its teeth mesh with only 32 of the follower 1's 64 teeth (i.e. follower 1 only turns half way round). Driver 2 is fixed to follower 1 so it only turns half way round too. A half turn will mean only 12 of its teeth will mesh with the 72 teeth of the final gear, i.e. the final gear will not move much at all.

If we were to work out *exactly* how much bigger than the driver the final gear would have to be in a simple gear train to turn this slowly, we would have the overall gear ratio. We could then use the overall gear ratio to work out how fast the final gear turns. If it acts as if it were 10 times as big, it would turn at one tenth the speed. The problem has to be worked in two separate parts.

a Overall gear ratio:

CHOOSE WHAT TO DO

Driver 1 and follower 1 are a pair; driver 2 and follower 2 are a pair. The first simple gear ratio is $\frac{1}{2}$. The second is $\frac{1}{3}$ So the overall gear ratio would have to be $\frac{1}{2}$ of $\frac{1}{3}$.

WORK OUT AN ANSWER

$\frac{1}{2}$ of $\frac{1}{3}$ means $\frac{1}{2} \times \frac{1}{3}$. The answer will have to be a fraction smaller than $\frac{1}{3}$.

LOOK BACK

An overall gear ratio of $\frac{1}{6}$ means that the final gear acts as if it were six times bigger than the driver in a simple gear train. This sounds right. The point of having a compound gear train is that you don't have to use such huge gears.

b Speed of the final gear:

CHOOSE WHAT TO DO

If the final gear acts as if it were six times as big it must turn at $\frac{1}{6}$ the speed, so you'll be finding $\frac{1}{6}$ or dividing by 6.

WORK OUT AN ANSWER

Use the formula. This is an easy division so you probably won't need your calculator.

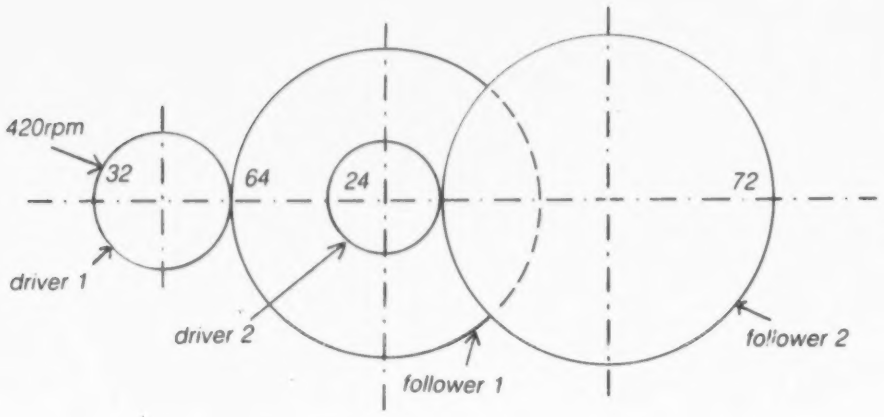
LOOK BACK

This is much slower than the driver, as expected.

Worked example 4:
Fitting and machining

In the compound gear train shown in the diagram find:

(a) the overall gear ratio and
(b) the speed of the final gear.



Driver 1 - 32 teeth *Driver 2 - 24 teeth*
Follower 1 - 64 teeth *Follower 2 - 72 teeth*

Speed of Driver 1 = 420rpm

(a) Overall gear ratio = simple gear ratios multiplied together

$$\text{Simple gear ratio} = \frac{\text{no. of teeth in driver}}{\text{no. teeth in follower}}$$

$$\text{Gear ratio 1} = \frac{32}{64}$$

$$= \frac{1}{2}$$

$$\text{Gear ratio 2} = \frac{24}{72}$$

$$= \frac{1}{3}$$

$$\text{Overall gear ratio} = \frac{1}{2} \times \frac{1}{3}$$

$$= \frac{1}{6}$$

(b) Speed of final gear = Speed of driver 1 × overall gear ratio

$$= 420\text{rpm} \times \frac{1}{6}$$

$$= 70\text{rpm}$$

The final gear turns at 70rpm.

Worked example 5:
Notes on the problem-solving steps

GET TO KNOW THE PROBLEM

Sales are always 100%. All the parts must add up to sales Writing down what you are given will help you see what has to be done. The four parts that make up sales in this problem are labour, overheads, food and profit. You are given all three of them as either an amount or a percentage. It will help if you write all of the ones you know as an amount. If they were also worked out as a percentage, that would be a double check.

CHOOSE WHAT TO DO

You can work out the total dollars and three of the parts, but profits and overheads are only a percentage. They will have to be changed to dollars. If you add up the parts and subtract them from the total, it should give you the food costs.

WORK OUT THE ANSWER

There are five bits to work out

- Total takings
- Profits
- Overheads
- Profits + overheads + labour
- Food

Make a rough estimate of each first and then use a calculator to work them out exactly.

LOOK BACK

The answer is fairly close to the estimate. Check that all four parts add up to $2250 — they do! Comparing food costs to takings gives about $^1/_3$ and that's a good rule of thumb.

Worked example 5: Commercial cookery

A caterer was asked to plan a dinner for 50 guests, each paying $45. The labour costs were estimated to be $780. Overheads were reckoned as 22% of sales. How much would the caterer spend on food if she/he wanted to make a profit of 10% sales?

Food = sales – (profit + overheads + labour)

Calculations		$	$	Estimate
Sales	50 × $45	225	2250	50 × 50 = 2500
Profit	10% of 2250 = 2250 ÷ 10			250
Overheads	22% of 2250 = 2250 ÷ 100 × 22			$\frac{1}{4}$ × 2500 = 650
	or 2250 × 22 %	495		
Labour		780		800
Sub-total			1500	1700
Food	Sales – sub-total = 2250 – 1500		750	2500 – 1700 = 800

Food cost = $750

The caterer would have $750 to spend on food.